A Deterministic PTAS for the Algebraic Rank of Bounded Degree Polynomials

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Introduction

Problem

Given an $n \times n$ matrix Q with polynomials as entries, compute $\text{rank}(Q)$. That is, given $Q \in \mathbb{F}[x_1, x_2, \ldots, x_m]^{n \times n}$, compute $\text{rank}(Q)$ over the rational function field $\mathbb{F}(x_1, x_2, \ldots, x_m)$.

- If $\mathbb{F}$ is large enough then substituting random values for x_i's is enough (Schwartz-Zippel lemma).
- Goal: deterministic algorithms for computing $\text{rank}(Q)$.

PIT of ABPs and Algebraic Rank

- Polynomial identity testing of algebraic branching programs reduces to it.
  - Thus exact deterministic algorithms are hard.
  - Computing a approximation of $\mathrm{rank}(Q)$?
- Computing the Transcendence degree of a set of polynomials (Jacobian criterion).

Contribution

Problem

Given $Q \in \mathbb{F}[x_1, x_2, \ldots, x_m]^{n \times n}$ with Q_{ij} homogeneous, $\deg(Q_{ij}) \leq d$ and $0 < \epsilon < 1$, find λ_i's such that:

$$\operatorname{rank}(Q(\lambda_1, \lambda_2, \ldots, \lambda_m)) \geq (1 - \epsilon) \operatorname{rank}(Q)$$

.

Theorem

There exists a deterministic algorithm which solves above problem in time $O\left((nmd)^{O\left(\frac{d^2}{\epsilon}\right)} \cdot M(n)\right)$, $M(n)$ is the time needed to compute the rank of an $n \times n$ matrix over $\mathbb{F}$.

History

- When $d = 1$, it is the problem of computing the commutative rank of matrix spaces.
- Computing the non-commutative rank of matrix spaces in deterministic polynomial time (Garg, Oliveira, Gurvits, Wigderson in FOCS 2016).
- commutative rank $\leq$ non-commutative rank $\leq 2 \cdot$(commutative rank).

History

- Thus $\frac{1}{2}$-approximation in deterministic polynomial time for $d = 1$.
- For $d = 1$, $(1 - \epsilon)$ approximation in time $O\left((nm)^{O\left(\frac{1}{\epsilon}\right)} \cdot M(n)\right)$ (Bläser, Jindal, Pandey in CCC 2017).

Main Idea

- Suppose already found $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that

$$\mathrm{rank}(Q(\lambda_1, \lambda_2, \ldots, \lambda_m)) = r.$$

- Want to find $(\mu_1, \mu_2, \ldots, \mu_m)$ such that

$$\mathrm{rank}(Q(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_m + \mu_m)) > r.$$

- Assume $Q(\lambda_1, \lambda_2, \ldots, \lambda_m) = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

d=2

$$\begin{aligned} Q(\lambda_1 + x_1, \lambda_2 + x_2, \ldots, \lambda_m + x_m) = {} & Q(\lambda_1, \lambda_2, \ldots, \lambda_m) \\ & + L(x_1, x_2, \ldots, x_m) \\ & + Q(x_1, x_2, \ldots, x_m) \end{aligned}$$

- L has homogeneous degree 1 entries and Q has homogeneous degree 2 entries.
- Want to find an assignment $x_i = \mu_i$ such that there is a non-zero $(r+1) \times (r+1)$ minor of $Q(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_m + \mu_m)$.

Special PIT

- If a polynomial has a non-zero monomials of low degree then we can do its PIT "easily."

Definition

$F_{m,d,\ell} \stackrel{\text{def}}{=\!=} \{f \in \mathbb{F}[x_1, x_2, \ldots, x_m] \mid \deg(F) \leq d$ and f has a non-zero monomial of degree atmost $\ell\}$.

Lemma

We can construct a hitting set $H_{m,d,\ell} \subseteq \mathbb{F}^m$ of size $O((m(d+1))^{\ell})$ for $F_{m,d,\ell}$.

d=2

$$Q(\lambda_1 + x_1, \lambda_2 + x_2, \ldots, \lambda_m + x_m) = \begin{bmatrix} I_r + L_{11} + Q_{11} & L_{12} + Q_{12} \\ L_{21} + Q_{21} & L_{22} + Q_{22} \end{bmatrix}$$

- Here L_{ij} have homogeneous degree 1 entries and Q_{ij} have homogeneous degree 2 entries.
- If all the $(r+1) \times (r+1)$ minors do not have any degree zero, one and two monomials then $Q_{22} = L_{21} L_{12}$.
    - Implies that $\text{rank}(Q) \leq 3r \implies$ already $\frac{1}{3}$-approximation for $\text{rank}(Q)$.
    - Otherwise easy to find desired $x_i = \mu_i$.

d=2

- Analyze the higher degree monomials of all the $(r+1) \times (r+1)$ minors.
- If all the $(r+1) \times (r+1)$ minors do not have any $\leq k$ degree monomials.
    - Implies $\text{rank}(Q) \leq r(1+\frac{2}{k-1})$.
- Higher k gives better approximation.

General Idea

- Same idea work for general d as well.
- Adjoint helps in analyzing the higher degree monomials of relevant $(r+1) \times (r+1)$ minors.
- If all the $(r+1) \times (r+1)$ minors do not have any $\leq k$ degree monomials.
  - Implies $\text{rank}(Q) \leq r(1 + \frac{d}{k-d+1})$.s
- Higher k gives better approximation.

Thanks

Thank you for your attention!