PosSLP and Sum of Squares


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Decision problems on Integers

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Example

Given $a, b, c, n \in \mathbb{N}$, $N := a^n + b^n - c^n$. Decide if:

- N is zero (Fermat's Last Theorem) ?
- N is positive?

Arithmetic circuits and SLPs

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Definition (Arithmetic circuit)

An arithmetic circuit is a directed acyclic graph whose inputs are constants $0, 1$ or indeterminates $x_1, x_2, \ldots, x_n$. Internal nodes are operations $+, -, \times$.

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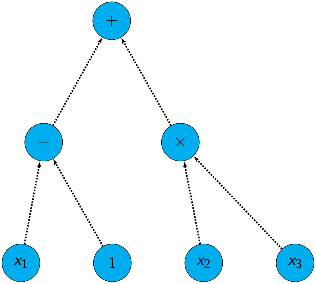
Definition (SLP)

A straight-line program (SLP) is a sequence of instructions for evaluation of an arithmetic circuit.

- SLPs and arithmetic circuits are used interchangeably.

Example



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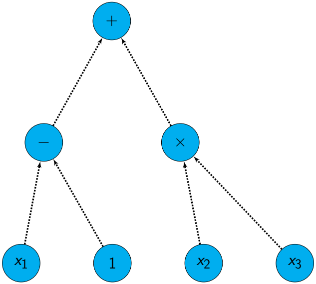

- This circuit computes the polynomial $(x_1 - 1) + x_2 x_3$.

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- Such a SLP P is sequence of integers $(b_0, b_1, b_2, \ldots, b_\ell)$ with
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Remark

We **cannot** simply compute b_ℓ.

Sum of Square Roots Problem

Problem (SSR)

Given $S = \sum_{i=1}^{n} \delta_i \sqrt{a_i}$*, with* $\delta_i \in \{+1, -1\}$ *and* $a_i \in \mathbb{N}$*, decide if* $S > 0$.

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SSR $\leq_{\mathrm{P}}$ PosSLP.

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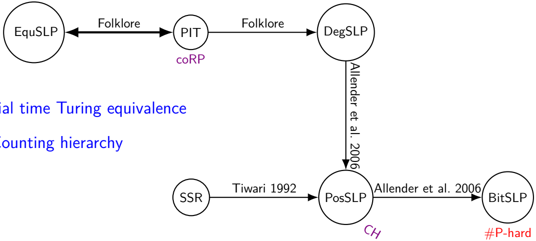
Proof.

A lower bound on $|S|$. Newton iteration: If $x_0 = a$, $x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right)$ then $x_i \to \sqrt{a}$. □

Complexity landscape of PosSLP



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CH = Counting hierarchy

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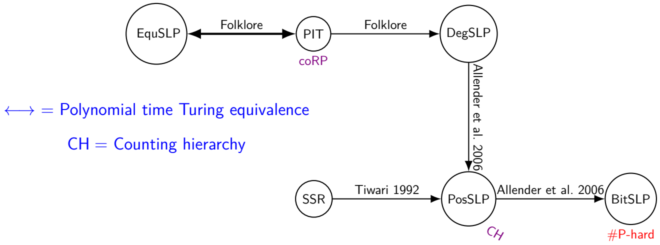

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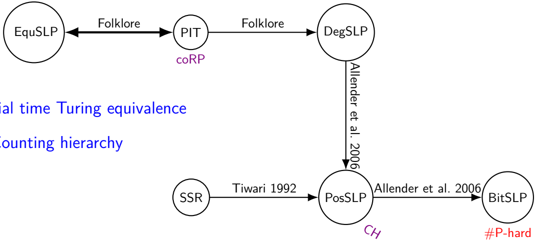

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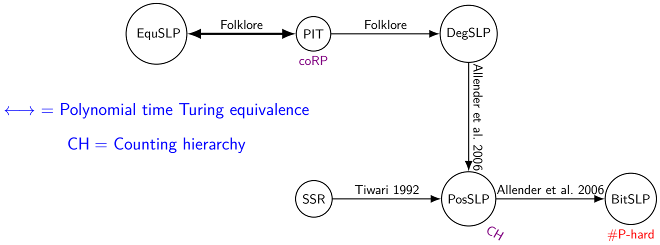

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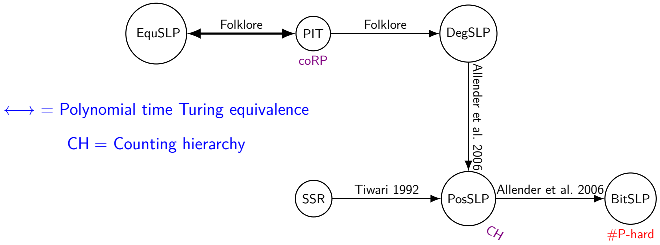

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- BitSLP: Given a SLP computing $N \in \mathbb{Z}$ and $i \in \mathbb{N}$, decide if i^{th} bit of N is 1.

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Theorem ((Bürgisser and Jindal 2024))

If a constructive variant of the ***radical conjecture*** *of (Dutta, Saxena, and Sinhababu 2018) is true and* PosSLP $\in$ BPP *then* NP $\subseteq$ BPP.

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- There exist integer sequences where $\tau_+(n) > \tau(n)$ (Jindal and Saranurak 2012).

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- So PosSLP is same as:
  - First check if $N = 0$ using EquSLP.
  - Given non-zero N (as SLP) decide if $\exists a, b, c, d \in \mathbb{N}$ such that $N = a^2 + b^2 + c^2 + d^2$.

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- How about shorter Sum of Squares certificates?

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- $n \in \mathbb{N}$ is 3SoS if it can be expressed as the sum of three squares (of integers).
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Lemma

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Proof.

Suppose $M = N^2$. If $M \in \mathbb{Z}_+$ then $7M^4$ not a 3SoS. □

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Proof.

Given SLP for N, first check if $N \in \{0, -1, -2\}$. Assume $N \notin \{0, -1, -2\}$. Check if N is a 3SoS. Check if $N+2$ is a 3SoS. $\square$

3SoSSLP and Div2SLP

Problem (Div2SLP)

Given $N \in \mathbb{Z}$ by SLP, and $\ell \in \mathbb{N}$, decide if 2^{ℓ} divides $|N|$.

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Proof.

To decide if N is 3SoS, first check if $N > 0$. Then use Div2SLP oracle to check the 3SoS condition. □

2SoSSLP

Theorem (Gauss 1801; Jacobi 1829)

An integer $n > 1$ is not 2SoS if and only if the prime-power decomposition of n contains a prime of the form $4k + 3$ with an odd power.

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Theorem

If Generalized Cramér conjecture is true then PosSLP $\in \mathrm{NP}^{\mathrm{2SoSSLP}}$.

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Proof.

To decide if $N = a^2$, choose a random prime p and decide if N mod p is a square in $\mathbb{F}_p$. □

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Proof.

Pick a random $t \in \mathbb{Z}$ and decide if $f(t)$ is a square using algorithm for SquSLP. This works by using an effective variant of Hilbert's irreducibility theorem. $\square$

Summary of Complexity reductions

Problem (OrdSLP)

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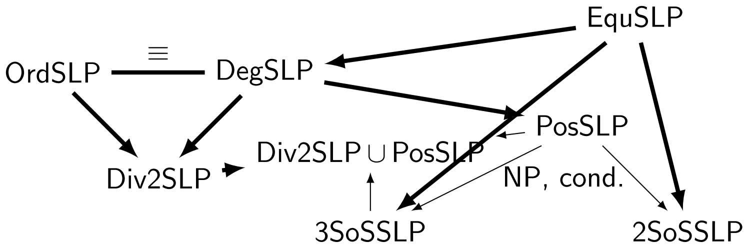

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- Efficient algorithms for special cases of PosSLP.

Thanks for your attention! Any questions?

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