On the Hardness of PosSLP

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January 8, 2024

SODA 2024, Alexandria, Virginia, USA

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Motivation: Numerical stable algorithms

- An algorithm (represented as function f) with $f\colon \mathbb{R}^n \to \mathbb{R}$
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- It is reasonable to assume that f can be approximated using polynomials

Arithmetic circuits and SLPs

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Arithmetic circuits and SLPs

Definition (Arithmetic circuit)

An arithmetic circuit is a directed acyclic graph whose inputs are constants $0, 1$ or indeterminates $x_1, x_2, \ldots, x_n$. Internal nodes are operations $+, -, \times, \div$.

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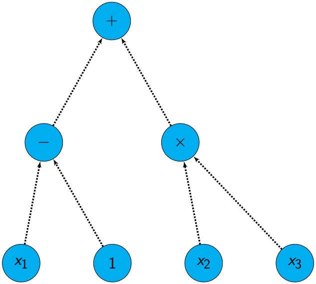
Definition (SLP)

A straight-line program (SLP) is a sequence of instructions for evaluation of an arithmetic circuit.

- SLPs and arithmetic circuits are used interchangeably.

Example



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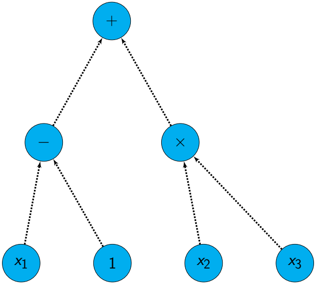

- This circuit computes the polynomial $(x_1 - 1) + x_2 x_3$

Floating point representations

- For any non-zero $u \in \mathbb{R}$, there exists unique $u' \in \mathbb{R}, m \in \mathbb{Z}$ with $\frac{1}{2} \leq |u'| < 1$ such that $u = u'2^m$

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- This pair (v, m) is a floating point approximation of u with k significant bits

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Task of a numerical analyst

- Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as inputs, approximate $f(x)$

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Problem (Generic task of numerical computation (GTNC))

Given a SLP P with indeterminates $x_1, x_2, \ldots, x_n$, floating point numbers $a_1, a_2, \ldots, a_n$ and an integer k in unary, compute a floating point approximation of $P(a_1, a_2, \ldots, a_n)$ with k significant bits.

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Given a division-free SLP P without indeterminates, decide if the integer N computed by P is positive.

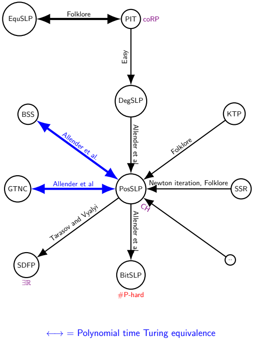
- Such an SLP P is sequence of integers $(b_0, b_1, b_2, \ldots, b_\ell)$ with $b_0 = 1$ and for all $1 \leq i \leq \ell$, $b_i = b_j \circ_i b_k$, where $\circ_i \in \{+, -, *\}$ and $j, k < i$
- Integer computed by P is b_ℓ, Size of P is ℓ

Connection to numerical analysis

Theorem ((Allender et al. 2006))

GTNC *is polynomial time Turing equivalent to* PosSLP.

Complexity landscape of PosSLP



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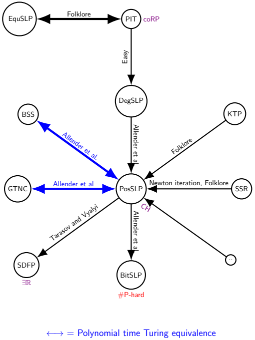

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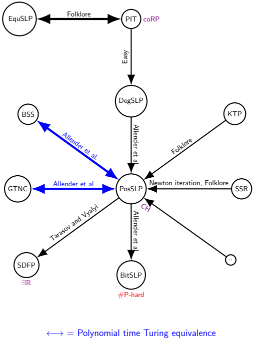

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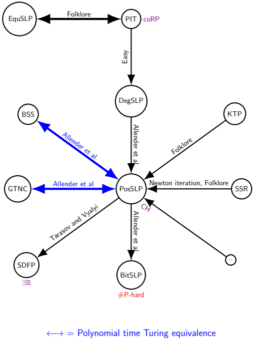

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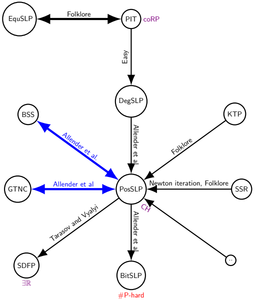

$\longleftrightarrow$ = Polynomial time Turing equivalence

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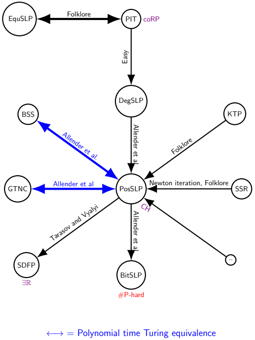

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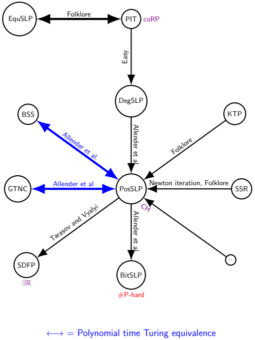

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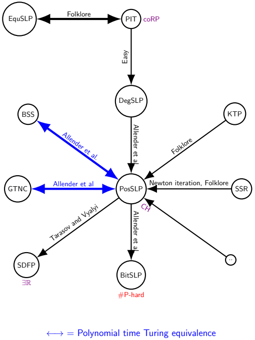

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- CH: Counting hierarchy
- $\exists\mathbb{R}$: decide if a given semialgebraic set is non empty

Upper bounds for PosSLP

Theorem ((Allender et al. 2006))

PosSLP $\in$ CH, *here* CH *is the counting hierarchy.*

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- There exist integer sequences where $\tau_+(n) > \tau(n)$ (Jindal and Saranurak 2012)

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Lower bounds for PosSLP

- Unfortunately, nontrivial lower bounds for PosSLP remain unknown

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Theorem (This paper)

If a constructive variant of the **radical conjecture** *is true and* PosSLP $\in$ BPP *then* NP $\subseteq$ BPP.

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  - 3SAT formula $\phi \longrightarrow$ An arithmetic circuit computing a polynomial f_ϕ such that ϕ is satisfiable iff f_ϕ has a real root
  - All the real roots (if any) of f_ϕ are in $(-1, 1)$

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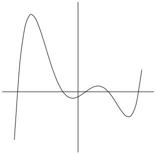
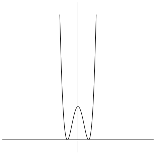
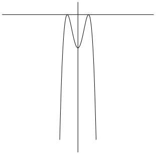

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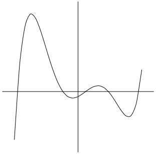
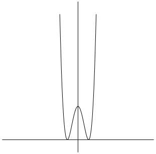
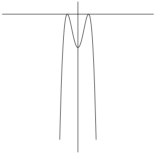

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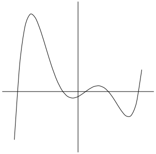
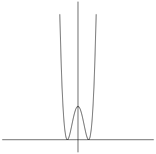
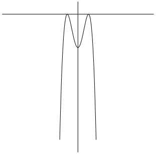

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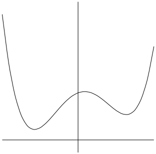
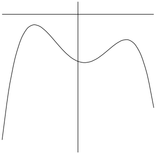

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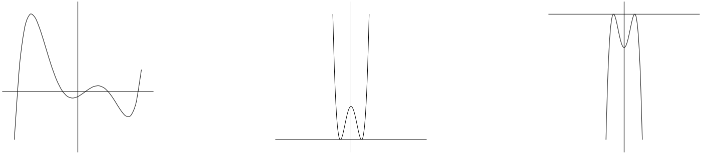

- If ϕ is not satisfiable then f_ϕ "should" look like:

- Pick a random rational q in $(-1, 1)$, check if $f_\phi(1)$ and $f_\phi(q)$ have different signs, i.e, $(-1)f_\phi(1)f_\phi(q) > 0$

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Challenges

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- What if f_ϕ has roots only of even multiplicity? $\rightsquigarrow$ **Radical conjecture**
- Even if f_ϕ has roots only of odd multiplicity, how likely is it that $f_\phi(1)$ and $f_\phi(q)$ have different signs? $\rightsquigarrow$ **UniqueSAT**

Radical conjecture

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Conjecture ((Dutta, Saxena, and Sinhababu 2022), Radical conjecture)

For univariate $f \in \mathbb{Z}[x]$, $\tau(\mathrm{rad}(f)) \leq \mathrm{poly}(\tau(f))$.

- It implies $\tau(\mathrm{rad}(f_\phi)) \leq \mathrm{poly}(n)$, $n =$ number of literals in ϕ

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UniqueSAT

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- Under randomized polynomial time reductions, ϕ can be assumed to have a unique satisfying assignment (Valiant and Vazirani 1986)

Theorem (This paper)

If ϕ has a unique satisfying assignment then $\mathrm{rad}(f_\phi)(1)$ and $\mathrm{rad}(f_\phi)(q)$ have different signs with probability at least $\frac{1}{4\pi}$, where q is a random rational in $(-1, 1)$.

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Lower Bound

Lower Bound

Theorem (This paper)

If a constructive variant of the radical conjecture is true and PosSLP $\in$ BPP *then* NP $\subseteq$ BPP.

Future research directions

- Radical conjecture is a strong assumption, can we do without it?

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- Radical conjecture is a strong assumption, can we do without it?
- Special cases of PosSLP:
  - Koiran's trinomial sign problem (Koiran 2019)
  - Sum of square roots problem
  - Decide if $a^n + b^n - c^n > 0$
  - Many more.....

Thanks for your attention! Any questions?

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