Geometric complexity theory for product-plus-power

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Abstract

According to Kumar's recent surprising result (ToCT'20), a small border Waring rank implies that the polynomial can be approximated as a sum of a constant and a small product of linear polynomials. We prove the converse of Kumar's result and establish a tight connection between border Waring rank and the model of computation in Kumar's result. In this way, we obtain a new formulation of border Waring rank, up to a factor of the degree.

We connect this new formulation to the orbit closure problem of the product-plus-power polynomial. We study this orbit closure from two directions:

1. We deborder this orbit closure and some related orbit closures, i.e., prove all points in the orbit closure have small non-border algebraic branching programs.

2. We fully implement the geometric complexity theory approach against the power sum by generalizing the ideas of Ikenmeyer-Kandasamy (STOC'20) to this new orbit closure. In this way, we obtain new multiplicity obstructions that are constructed from just the symmetries of the polynomials.

Keywords: border complexity, Waring rank, geometric complexity theory, Newton identities
2020 Math. Subj. Class.: 68Q17, 05E05, 05E10, 22E60

1 Introduction

Waring rank is an important complexity measure for homogeneous polynomials, classically studied in geometry and invariant theory. In complexity theory, it defines a model of computation also known as the homogeneous diagonal depth-3 circuits, see e.g. [Sax08]. In addition, the *matrix multiplication exponent*, a fundamental constant in computational complexity, can be described in terms of the Waring rank of a particular family of polynomials, which are a symmetrized version of the matrix multiplication tensors [CHI+18]. In this work, we closely relate Waring rank and its *border* version to a new measure that we call *border Kumar complexity*, inspired by the recent work of Kumar [Kum20].

The *Waring rank* of a homogeneous degree d polynomial f, denoted $\mathsf{WR}(f)$, is the *smallest* r such that there exist homogeneous linear polynomials (also called 'linear forms') $\ell_1, \ldots, \ell_r$, with $f = \sum_{i \in [r]} \ell_i^d$. This is a natural generalization of the rank of a symmetric matrix, but when $d \geq 3$, a sequence of homogeneous polynomials of Waring rank r may converge to a polynomial of Waring rank strictly larger than r. The *border Waring rank* of f, denoted $\underline{\mathsf{WR}}(f)$, is the smallest r such that f can be written as the limit of a sequence of polynomials f_ϵ with $\mathsf{WR}(f_\epsilon) = r$.

Following [Kum20], we introduce another measure of complexity for homogeneous polynomials: given a homogeneous f of degree d, the *Kumar complexity* of f, denoted $\mathsf{Kc}(f)$, is the smallest m such that there exists a constant $\alpha \in \mathbb{C}$ and homogeneous linear polynomials $\ell_1, \ldots, \ell_m$ with the property that

$$f = \alpha\left(\prod_{i=1}^{m}(1 + \ell_i) - 1\right). \tag{1.※}$$

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For instance, given a linear form ℓ, we see that $\mathsf{Kc}(\ell^d) = d$, because $\ell^d = \prod_{j=1}^{d}(1 + \zeta^j \ell) - 1$, where ζ is a primitive d-th root of unity. However, not all polynomials have finite Kumar complexity: for example, it is easy to see that $x_1 \cdots x_n$ cannot be expressed as in (1.✳). The *border Kumar complexity* of f, denoted $\underline{\mathsf{Kc}}(f)$, is the smallest m such that

$$f = \lim_{\epsilon \to 0} \alpha(\epsilon)\big(\textstyle\prod_{i=1}^{m}(1 + \ell_i(\epsilon)) - 1 \big),$$

for $\alpha(\epsilon) \in \mathbb{C}[\epsilon^{\pm 1}]$, and linear forms $\ell_i \in \mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]_1$. Alternatively, one can define $\alpha \in \mathbb{C}(\epsilon)$ and $\ell_i \in \mathbb{C}(\epsilon)[\boldsymbol{x}]_1$, which is equivalent, as can be seen analogously to [BCS97, Lem. 15.22]. We will only be working over $\mathbb{C}[\epsilon^{\pm 1}]$.

Kumar [Kum20] proved that $\underline{\mathsf{Kc}}(f)$ is finite for every homogeneous polynomial f. More precisely, he proved $\underline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \mathsf{WR}(f)$, and it is easy to see that in fact $\underline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \underline{\mathsf{WR}}(f)$. This inequality is not tight: for instance the square-free monomial of degree n can be written as $x_1 \cdots x_n = \lim_{\epsilon \to 0} \epsilon^n \big(\prod_i (1 + \frac{1}{\epsilon} x_i) - 1 \big)$ which shows $\underline{\mathsf{Kc}}(x_1 \cdots x_n) = n$, whereas it is a classical fact that $\underline{\mathsf{WR}}(x_1 \cdots x_n) \geq \binom{n}{\lfloor n/2 \rfloor}$, which is exponentially large as a function of n [1]. We develop a border version of the Newton identities and use it to prove a converse Kumar's result, showing that completely reducible forms are essentially the *only* example where Kumar's inequality is far from being tight:

1.1 Theorem (Converse of Kumar's theorem). *Let f be a homogeneous polynomial. Then either $\underline{\mathsf{WR}}(f) \leq \underline{\mathsf{Kc}}(f)$ or f is a product of linear forms.*

One important consequence of the close relationship between border Waring rank and Kumar complexity is a new characterization of the matrix multiplication exponent. The matrix multiplication exponent is defined as

$$\omega = \inf\{\tau : \text{ two } n \times n \text{ matrices can be multiplied using } O(n^\tau) \text{ scalar multiplications}\}.$$

This fundamental constant can be defined in terms of the tensor rank and the tensor border rank of the matrix multiplication tensor [Str69, BCLR79]. The results of [CHI+18] show that $\omega = \lim_{n \to \infty} \log_n \underline{\mathsf{WR}}(\mathrm{trace}(X_n^3))$, where $X_n = (x_{ij})_{i,j=1,\dots,n}$ is a matrix of variables, so $\mathrm{trace}(X_n^3)$ is a homogeneous degree 3 polynomial in n^2 variables. This result, together with Theorem 1.1, implies the following

1.2 Corollary. *The exponent of matrix multiplication is characterized as*

$$\omega = \lim_{n \to \infty} \log_n \underline{\mathsf{Kc}}(\mathrm{trace}(X_n^3)).$$

Theorem 1.1, together with Corollary 1.2, serves as motivation to study upper and lower bounds methods for $\underline{\mathsf{Kc}}$. In order to study Kumar's model from a geometric point of view, we introduce a homogenized version of it, that we call *product-plus-power model*. We provide several results on this model:

- We prove *debordering results*, providing strong upper bounds to the border Waring rank of homogeneous polynomials in terms of their complexity in the product-plus-power model, see Section 1.2.

- We implement the geometric complexity theory program (GCT), providing an infinite family of *obstructions*, see Section 1.3.

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[1] We remark that determining the exact value of $\underline{\mathsf{WR}}(x_1 \cdots x_n)$ is an open problem. It is known that $\mathsf{WR}(x_1 \cdots x_n) = 2^{n-1}$ [CCG12]. The same result on cactus rank (a scheme-theoretic version of Waring rank) is proved in [RS11]. There are (at least) two incorrect/incomplete proofs available online of the same result for border rank: the early versions of [Oed19], and the first version of [CGO19]. A discussion on the gaps in the proofs is available in the first version of [BB19, Sec 6.1]

1.1 The product-plus-power model

Homogeneous models of algebraic computation and their border versions are conveniently described using the notions of restrictions and degenerations of homogeneous polynomials. Let $f(\mathbf{x}), g(\mathbf{x})$ be homogeneous polynomials of degree d in n variables $x_1, \ldots, x_n$. We say that g is a *restriction* of f if there exist linear forms $\ell_1(\mathbf{x}), \ldots, \ell_n(\mathbf{x})$ such that $g(\mathbf{x}) = f(\ell_1(\mathbf{x}), \ldots, \ell_n(\mathbf{x}))$; in this case, write $g \leq f$. Equivalently, regarding f, g as polynomial functions on $\mathbb{C}^n$, we have that $g \leq f$ is there exist a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ such that $g(\mathbf{x}) = f(A^t \mathbf{x})$. Here A^t is the transpose of the matrix A.

We say that g is a *degeneration* of f is there exist linear forms $\ell_1(\mathbf{x}), \ldots, \ell_n(\mathbf{x})$, depending rationally on a parameter ϵ, such that $g(\mathbf{x}) = \lim_{\epsilon \to 0} f(\ell_1(\mathbf{x}), \ldots, \ell_n(\mathbf{x}))$; in this case, write $g \trianglelefteq f$. Equivalently, $g(\mathbf{x}) = \lim_{\epsilon \to 0} f(A_\epsilon \mathbf{x})$ where A_ϵ is a linear map $A_\epsilon : \mathbb{C}^n \to \mathbb{C}^n$ depending on ϵ. It is a classical fact that the entries of A_ϵ, or equivalently the coefficients of $\ell_i(\mathbf{x})$, can be taken in the ring of Laurent polynomials $\mathbb{C}[\epsilon^{\pm 1}]$, see e.g. [BCS97, Section 20.6]; in particular, the polynomial $f(A_\epsilon \mathbf{x})$ is an element of $\mathbb{C}[\epsilon^{\pm 1}, x_1, \ldots, x_n]$.

The number of variables does not play a role, and we will occasionally use the slightly larger ring $\mathbb{C}[x_0, x_1, \ldots, x_n]$ for notational convenience.

Define the product-plus-power polynomial $P_{1,1}^{[d]} = x_1 \cdots x_d + x_0^d \in \mathbb{C}[x_0, x_1, \ldots, x_n]_d$. Given a homogeneous degree d polynomial f in the variables $x_1, \ldots, x_n$ (without x_0), it is clear that if $\underline{\mathsf{Kc}}(f) \leq m$ then $x_0^{m-d} f$ is a degeneration of $P_{1,1}^{[m]}$. Therefore, lower bounds on the smallest possible m such that $x_0^{m-d} f \trianglelefteq P_{1,1}^{[m]}$ provide lower bounds on $\underline{\mathsf{Kc}}(f)$ and hence on $\underline{\mathsf{WR}}$.

It is unclear to what extent the product-plus-power model is stronger than Kumar's model, so we provide *debordering results* for the product-plus-power model, see Section 1.2. We establish that this model is not much stronger, hence lower bounds are expected to exist. To find such lower bounds on Kumar's complexity, we implement the geometric complexity theory approach for the product-plus-power model, see Section 1.3.

1.2 Debordering the product-plus-power model

Border Waring rank, and border Kumar complexity are *border* measures of complexity, that is they are defined in terms of degeneration. It is often unclear what the gap between a non-border complexity measure and its border complexity measure can be. Lower bounds of a border complexity measure in terms of a non-border measure are commonly called *debordering results* and they guarantee that introducing the degeneration procedure does not make a model of computation much stronger. We provide debordering results for the product-plus-power model and some generalizations.

We record one important debordering result for border Waring rank, in terms of the algebraic branching program width. Given a homogeneous polynomial f of degree d, the *algebraic branching program width* of f, denoted $\mathsf{abpw}(f)$, is the smallest w such that f can be expressed as a product $f(\mathbf{x}) = A_1(\mathbf{x}) \cdots A_d(\mathbf{x})$ of matrices whose entries are linear forms, with A_1 of size $1 \times w$, A_k of size $w \times w$ for $k = 2, \ldots, d-1$ and A_d of size $w \times 1$. It is known that $\mathsf{abpw}(f) \leq \underline{\mathsf{WR}}(f)$ [For16, BDI21], and further, these measures can be *exponentially* far (for e.g., see [CKW11]). In particular, Theorem 1.1 can be interpreted as a debordering result for Kumar's complexity, showing $\mathsf{abpw}(f) \leq \underline{\mathsf{Kc}}(f)$.

Similarly, other debordering results for border Waring rank, such as [DGI+24, Theorem 1], and more generally classification results for secant varieties as in [BL14, Bal17, BB13], can be reinterpreted in terms of Kumar's complexity via Theorem 1.1.

We obtain the following result for degenerations of the product-plus-power polynomial $P_{1,1}^{[d]}$:

1.3 Theorem (Debordering product-plus-power). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$. If $f \trianglelefteq P_{1,1}^{[d]}$, then*

(i) either $f \leq P_{1,1}^{[d]}$

(ii) or $\underline{\mathsf{WR}}(f) \leq O(d^5)$.

Also, we will obtain debordering results for a variant of the product-plus-power polynomial, that is the product-plus-two-powers polynomial $P_{1,2}^{[d]} = x_1 \cdots x_d + x_0^d + x_{d+1}^d$, as well. These debordering results are more fine-grained because the previous techniques would show that $\mathsf{abpw}(f)$ is polynomially-bounded [DDS21]. In Section 3.2, we show exponential lower bounds for the same (see Theorem 3.15-Theorem 3.16).

1.3 The GCT program

Mulmuley and Sohoni introduced the geometric complexity theory (GCT) program [MS01, MS08] as an approach to Valiant's determinant vs permanent problem. The approach aims to determine "representation theoretic obstructions" in order to prove lower bounds of certain complexity measures, typically by proving the impossibility of a degeneration. Basic background knowledge on the representation theory of the general linear group and the symmetric group can for example be found in [FH91] and [Ful97], and is especially valuable in Section 4.

The group GL_n has a natural action on $\mathbb{C}[x_1, \ldots, x_n]_d$ given by $g \cdot f = f \circ g^t$ or equivalently $g \cdot f(\mathbf{x}) = f(g^t \mathbf{x})$ for any $f \in \mathbb{C}[x_1, \ldots, x_n]_d$, $g \in \mathrm{GL}_n$, $\mathbf{x} \in \mathbb{C}^n$. Now, given two elements $f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_n]_d$, the condition that a polynomial f_1 degenerates to a polynomial f_2 is equivalent to the fact that $f_2 \in \overline{\mathrm{GL}_n \cdot f_1}$; here $\mathrm{GL}_n \cdot f_1 = \{g \cdot f_1 \mid g \in \mathrm{GL}_n\}$ denotes the orbit of f_1 under the group action and the overline indicates the closure, equivalently in the Zariski or the Euclidean topology of $\mathbb{C}[x_1, \ldots, x_n]_d$ [Kra84, AI.7.2 Folgerung]. Further, the membership statement $f_2 \in \overline{\mathrm{GL}_n \cdot f_1}$ is equivalent to the inclusion of orbit-closures $\overline{\mathrm{GL}_n \cdot f_2} \subseteq \overline{\mathrm{GL}_n \cdot f_1}$.

Let $\mathbb{C}\big[\mathbb{C}[x_1, \ldots, x_n]_d\big]$ be the ring of polynomials over $\mathbb{C}[x_1, \ldots, x_n]_d$: its elements are polynomials in the coefficients of the elements of $\mathbb{C}[x_1, \ldots, x_n]_d$. Given $f \in \mathbb{C}[x_1, \ldots, x_n]_d$, let $I(\overline{\mathrm{GL}_n \cdot f}) \subseteq \mathbb{C}\big[\mathbb{C}[x_1, \ldots, x_n]_d\big]$ denote the *vanishing ideal* of the algebraic variety $\overline{\mathrm{GL}_n \cdot f}$ and let $\mathbb{C}[\overline{\mathrm{GL}_n \cdot f}] = \mathbb{C}\big[\mathbb{C}[x_1, \ldots, x_n]_d\big]/I(\overline{\mathrm{GL}_n \cdot f})$ be the quotient ring, called the *coordinate ring* of $\overline{\mathrm{GL}_n \cdot f}$. It turns out that $\mathbb{C}[\overline{\mathrm{GL}_n \cdot f}]$ is a graded ring; denote by $\mathbb{C}[\overline{\mathrm{GL}_n \cdot f}]_D$ its degree D component. The condition $\overline{\mathrm{GL}_n \cdot f_2} \subseteq \overline{\mathrm{GL}_n \cdot f_1}$ is equivalent to the inclusion $I(\overline{\mathrm{GL}_n \cdot f_1}) \subseteq I(\overline{\mathrm{GL}_n \cdot f_2})$, yielding a surjection of graded rings $\mathbb{C}[\overline{\mathrm{GL}_n \cdot f_1}] \twoheadrightarrow \mathbb{C}[\overline{\mathrm{GL}_n \cdot f_2}]$. In summary, if f_1 degenerates to f_2, then for every degree D there is a surjective linear map

$$\mathbb{C}[\overline{\mathrm{GL}_n \cdot f_1}]_D \twoheadrightarrow \mathbb{C}[\overline{\mathrm{GL}_n \cdot f_2}]_D.$$

The group action of GL_n on $\mathbb{C}[x_1, \ldots, x_n]_d$ induces an action of GL_n on the coordinate rings via canonical pullback. This makes both $\mathbb{C}[\overline{\mathrm{GL}_n f_1}]_D$ and $\mathbb{C}[\overline{\mathrm{GL}_n f_2}]_D$ into finite dimensional representations of GL_n. The surjection is GL_n-equivariant, namely it commutes with the group action. The group GL_n is *reductive*, a condition that guarantees that both representations decompose into a direct sum of *irreducible representations*. For each partition λ, the *representation theoretic multiplicity* mult_λ counts the number of irreducible representations of type λ in such a decomposition; this number is independent of the decomposition. In the setting described here, λ is always a partition of Dd into at most n parts, and Schur's Lemma (see, e.g., [FH91, Lemma 1.17]) implies an inequality of representation theoretic multiplicities: $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n f_2}]_D) \geq \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n f_1}]_D)$.

The GCT program aims to prove obstructions to the existence of the surjection by determining violations of these inequalities. More precisely, the existence of any λ with $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n f_1}]_D) < \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n f_2}]_D)$ would guarantee $\overline{\mathrm{GL}_n f_2} \nsubseteq \overline{\mathrm{GL}_n f_1}$ and therefore show that f_2 is *not* a degeneration of f_1. This is called a *representation theoretic multiplicity obstruction*. If additionally $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n f_1}]_D) = 0$, then we call this an *occurrence obstruction*. It is a wide open question in geometric complexity theory in which situations orbit closure containment can be disproved by representation theoretic multiplicity obstructions. Very few examples are known [BI11, BI13a, IK20]. We determine multiplicity obstructions that show that the polynomial $x_0^d + \cdots + x_d^d$ is not a degeneration of $P_{1,1}^d$ (which for example could also be seen directly by comparing the dimensions of their orbit closures):

1.4 Theorem (New obstructions). *Let $d \geq 3$, and let $\lambda = (5d - 1, 1) + ((d + 1) \times (10d))$. These λ are representation theoretic multiplicity obstructions that show $x_0^d + \cdots + x_d^d \not\trianglelefteq P_{1,1}^{[d]}$:*

$$\mathrm{mult}_\lambda(\mathbb{C}\left[\overline{\mathrm{GL}_{d+1}\, P_{1,1}^{[d]}}\right]) \leq 4 < 5 = \mathrm{mult}_\lambda(\mathbb{C}\left[\overline{\mathrm{GL}_{d+1}(x_0^d + \cdots + x_d^d)}\right]).$$

This result extends the result of [IK20] from the product polynomial to product-plus-power by exhibiting multiplicity obstructions that are based entirely on the symmetries of the two polynomials; see Theorem 4.10 for details.

Indeed, from a representation-theoretic and a combinatorial point of view, the polynomial $P_{1,1}^{[d]} = \prod_{i \in [d]} x_i + x_0^d$ looks very similar to the well-studied product polynomial $P_{1,0}^{[d]} = \prod_{i \in [d]} x_i$, which was the object of several GCT papers [Kum15, BI17, DIP20, IK20]. A system of set-theoretic equations for its orbit-closure was known for over a century, due to Brill and Gordon [Gor94], and their representation theoretic structure has been recently described by Guan [Gua18]. We transfer as much as possible of the known theory for $P_{1,0}^{[d]}$ to the setting of $P_{1,1}^{[d]}$, in order to mimic the proof technique of [IK20].

In order to prove Theorem 4.10, we obtain several results about the more general polynomial $P_{r,s}^{[d]} := \sum_{i=1}^{r} \prod_{j=1}^{d} x_{ji} + \sum_{i=1}^{s} y_i^d$.

1. We determine the stabilizer of $P_{r,s}^{[d]}$ under the action of the group GL_{rd+s}, see Theorem 4.2.

2. We use the stabilizer to determine the representation theoretic structure of the coordinate ring of the orbit of $P_{1,1}^{[d]}$, which is achieved in Proposition 4.4.

3. We prove that $P_{r,s}^{[d]}$ is *polystable*, in the sense of invariant theory, see Proposition 4.5.

4. Polystability implies the existence of a fundamental invariant in the sense of [BI17]. In Proposition 4.7, in the case $P_{1,1}^{[d]}$, we show an interesting connection between the degree of this fundamental invariant and the famous Alon-Tarsi conjecture on Latin squares in combinatorics: the fundamental invariant appears in degree $d+1$ if and only if the Alon-Tarsi conjecture holds for d.

A *Latin square* is an $n \times n$ matrix with entries $1, \ldots, n$ such that each row and each column is a permutation. The *column sign* of a Latin square is the product of the signs of its column permutations. If n is odd, then there are exactly as many sign $+1$ Latin squares as sign -1 Latin squares, and a sign-reversing involution is obtained by switching the first two rows. The Alon-Tarsi conjecture states that for n even, the number of sign $+1$ and sign -1 Latin squares are different. The main references on the Alon-Tarsi conjecture are [AT92, Dri97, Gly10], where it is shown that the conjecture is true for $n = p \pm 1$ for all odd primes p.

1.4 Related work and context

Waring rank and border Waring rank are the objects of a long history of work in classical algebraic geometry and invariant theory, beginning in the nineteenth century [Cay45, Syl52, Cle61]. It is related to the classical study of secant varieties [Pal06, Ter11] and today it has strong connections to the study of the Gorenstein algebras [IK99, BB14] and the geometry of the Hilbert scheme of points [BB21, JM22].

In complexity theory, one is interested in the growth of a complexity parameter in a sequence of polynomials. In this context, we say that a p-family is a sequence of polynomials $(f_n)_{n \in \mathbb{N}}$ such that the degree and the number of variables of f_n are polynomially bounded as functions of n. The complexity classes VWaring and $\overline{\mathsf{VWaring}}$ consist of all p-families $(f_n)_{n \in \mathbb{N}}$ such that, respectively, $\mathrm{WR}(f_n)$ or $\underline{\mathrm{WR}}(f_n)$, are a polynomially bounded function of n. Important complexity classes include VBP and VNP, consisting of p-families with polynomially bounded determinantal complexity or permanental complexity, respectively. The *determinantal complexity* of a homogeneous polynomial

f of degree d in n variables $x_1, \ldots, x_n$ is the smallest N such that $x_0^{N-d} f$ is a restriction of the determinant polynomial $\det_N = \sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma \prod_{i=1}^N x_{i,\sigma(i)}$, which is a polynomial of degree N in N^2 variables. The *permanental complexity* is defined similarly, in terms of the permanent polynomial $\mathrm{per}_N = \sum_{\sigma \in \mathfrak{S}_N} \prod_{i=1}^N x_{i,\sigma(i)}$. It is known that $\mathsf{VBP} \subseteq \mathsf{VNP}$ [Val79, Tod92], and Valiant's determinant vs. permanent conjecture states that this inclusion is strict, that is $\mathsf{VNP} \not\subseteq \mathsf{VBP}$.

The border complexity classes are $\overline{\mathsf{VBP}}$ and $\overline{\mathsf{VNP}}$ are defined similarly, replacing the notion of restriction with the one of degeneration, that is allowing arbitrary close approximations of polynomials in terms of determinants or permanents, rather than an exact expression. These border complexity classes can be defined as topological closures as well, see [IS22]. A systematic study of border complexity classes was initiated in [MS01, Bü04]: the Mulmuley-Sohoni conjecture is a strengthening of Valiant's conjecture, and it predicts that $\mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}$.

It is wide open whether $\mathsf{VBP} = \overline{\mathsf{VBP}}$, hence it is unclear to what extent the Mulmuley-Sohoni conjecture is stronger than Valiant's conjecture. As a first step towards resolving this question, in [BIZ18] it is shown that border of width-2 algebraic branching programs define the same complexity class as algebraic formulas $\overline{\mathsf{VBP}_2} = \overline{\mathsf{VF}}$, over fields of characteristic $\neq 2$, and using [AW16] this implies that $\mathsf{VBP}_2 \subsetneq \overline{\mathsf{VBP}_2}$. Resolving similar problems of inclusion between a border complexity class and a non-border complexity class is the goal of the debordering techniques. For instance, the already mentioned inequality $\mathsf{abpw}(f) \leq \underline{\mathsf{WR}}(f)$ proves the inclusion $\overline{\mathsf{VWaring}} \subseteq \mathsf{VBP}$ [For16, BDI21]. In [DDS21, DS22], it is shown that $\overline{\Sigma^{[k]}\Pi\Sigma} \subsetneq \mathsf{VBP}$, where $\Sigma^{[k]}\Pi\Sigma$ is the class of p-families of polynomials which can be expressed as $\sum_{i=1}^k \prod_j \ell_{ij}$, for linear forms ℓ_{ij}. Very recently, [DIK$^+$24] showed that border width-2 ABPs over characteristic 2 can compute any algebraic formulas.

A major difficulty to achieve debordering results is that, in general, boundaries of orbits of algebraic groups may present strong geometric pathologies. In small cases, one can achieve *boundary classification* results. This was done in the case of border Waring rank at most 5 [BL14, BB13, Bal17], for the 3×3 determinant polynomial $\det_3$ [HL16], and partially for the binomial $P_{2,0}^{[d]} = x_1 \cdots x_d + x_{d+1} \cdots x_{2d}$ [Hü17, Ch. II.9]. There are however universal results [Kac80, Vak06, Jel20] hinting towards the difficulty of such a fine classification in general.

The GCT approach discussed in [Section 1.3](#) was introduced in [MS01, MS08] as a path toward the Mulmuley-Sohoni conjecture, and proposed to use occurrence obstructions to prove lower bounds on the determinantal complexity of the *padded* permanent $x_0^{N-n}\mathrm{per}_n$. The *no-go theorem* of [IP17, BIP19] proved that this is impossible by making use of the fact that [MS01, MS08] use the padded formulation of the Mulmuley-Sohoni conjecture. There exists *no such result* when the determinant is replaced, for instance, by the iterated matrix multiplication polynomial, so that the determinantal complexity is replaced by the algebraic branching program width. The potential of multiplicity obstructions is explored in [DIP20, IK20]: in particular, the GCT approach is used to prove that the power sum polynomial is not a product of homogeneous linear forms, although there are easier ways to prove this.

Friedman and McGuinness [FM19] give a survey about the Alon-Tarsi conjecture. The GCT result in [Kum15] is based on the conjecture. The conjecture has been generalized in numerous directions. [SW12] prove that Drisko's proof method cannot be used without modifications to prove the Alon-Tarsi conjecture. The same is true for results in [BI13b, BI17], some of which are based on generalizations or variants of the conjecture. The Polymath Project number 12 (https://polymathprojects.org) was devoted to the study of Rota's basis conjecture, which for even n is implied by the Alon-Tarsi conjecture, see [HR94]. [Alp17] proves an upper bound on the difference between the even and odd Latin squares.

2   Kumar's complexity and border Waring rank

In this section, we prove [Theorem 1.1](#), connecting Waring and border Waring rank to Kc-complexity and its variants. To obtain the result, we observe that $\underline{\mathsf{Kc}}$ expressions fall into three different cases, depending on whether the scalar $\alpha(\epsilon)$ in (1.⁂) converges to 0, converges to a nonzero constant, or diverges. We study these three cases independently. For the case where $\alpha(\epsilon)$ converges to zero, it is

easy to see that the resulting polynomial is a product of affine linear polynomials, see Lemma 2.8. For the case where $\alpha(\epsilon)$ converges to a nonzero value, we use the Newton Identities to obtain the desired lower bound given by the Waring rank, see Proposition 2.9. The case where $\alpha(\epsilon)$ diverges is the most interesting one as it is the one where cancellations occur in the limit; in this case the proof is obtained via a border version of Newton relations.

Let $e_k(x_1, \ldots, x_n)$ denotes the k-th **elementary symmetric** polynomial, defined by

$$e_k(x_1, \ldots, x_n) := \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k} \,;$$

Recall that by definition $e_0 = 1$. First, we record an immediate observation that will be useful throughout:

2.1 Remark. It is easy to observe that

$$\prod_{i=1}^{m}(1 + x_i) = \sum_{j=0}^{m} e_j(\mathbf{x})$$

where $\mathbf{x} = (x_1, \ldots, x_m)$. In particular, given a homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]_d$ of degree d, if $f = \alpha(\prod_{i=1}^{m}(1 + \ell_i) - 1)$ for homogeneous linear forms $\ell_1, \ldots, \ell_m$, then

$$e_j(\ell_1, \ldots, \ell_m) = 0 \quad \text{for all } j \neq d,$$
$$e_d(\ell_1, \ldots, \ell_m) = \tfrac{1}{\alpha} f.$$

Newton identities are a central tool in this section; they relate the elementary symmetric polynomials and the *power sum* polynomial, defined as $p_k(\boldsymbol{x}) := x_1^k + \cdots + x_n^k$.

2.2 Proposition (Newton Identities, see e.g. [Mac95], Section I.2)**.** *Let n, k be integers with $n \geq k \geq 1$. Then*

$$k \cdot e_k(x_1, \ldots, x_n) = \sum_{i \in [k]} (-1)^{i-1} e_{k-i}(x_1, \ldots, x_n) \cdot p_i(x_1, \ldots, x_n) \,.$$

In light of the Remark 2.1, the Kc model of computation is a sum of elementary symmetric polynomials. Shpilka [Shp02] studied a similar notion of circuit complexity called s_{sym}. For a polynomial f, $s_{sym}(f)$ is defined as the smallest m such that $f = e_d(\ell_1, \ell_2, \ldots, \ell_m)$ where $d = \deg(f)$ and ℓ_i are affine linear forms. It was proved in [Shp02] that $s_{sym}(f)$ is always finite, moreover several upper and lower bounds for $s_{sym}(f)$ were proven. The complexity Kc differs from $s_{sym}(f)$, as Kc can even be infinite. In fact, the only homogeneous polynomials with finite Kc-complexity are powers of linear forms, as the following lemma shows.

2.3 Lemma. *Let $f \in \mathbb{C}[\mathbf{x}]_d$ be a homogeneous polynomial such that $\mathsf{Kc}(f) < \infty$. Then $\mathsf{Kc}(f) = d$ and f is a power of a linear form.*

Proof. If f is a homogeneous polynomial of degree d, then it is immediate that $\mathsf{Kc}(f) \geq \deg(f)$. Notice that for any linear form ℓ, we have $\ell^d = \prod_{i=1}^{d}(1 + \zeta^i \ell) - 1$ where ζ is a primitive d-th root of 1. This shows $\mathsf{Kc}(\ell^d) \leq d$, hence equality holds.

Assume $f \in \mathbb{C}[x]_d$ is a homogeneous polynomial with $\mathsf{Kc}(f) = m < \infty$. By definition $f = \alpha(\prod_{i=1}^{m}(1 + \ell_i) - 1)$ for some homogeneous linear forms $\ell_i \in \mathbb{C}[\mathbf{x}]$. Write $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m)$. By Remark 2.1, we have, $e_d(\boldsymbol{\ell}) = \frac{1}{\alpha} f$ and $e_j(\boldsymbol{\ell}) = 0$ for $j \neq d$.

First, observe $m = d$. Indeed, if $m > d$, we have $0 = e_m(\boldsymbol{\ell}) = \ell_1 \cdots \ell_m$, which implies $\ell_i = 0$ for some i, in contradiction with the minimality of m. Since $\mathsf{Kc}(f) \geq \deg(f)$, we deduce $m = d$.

Now we show that if $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$ satisfies $e_1(\boldsymbol{\ell}) = \cdots = e_{d-1}(\boldsymbol{\ell}) = 0$ then $e_d(\boldsymbol{\ell}) = (-1)^{d-1} \cdot \ell_d^d$; in particular, by unique factorization, all ℓ_i's are equal up to scaling. Write $\widehat{\boldsymbol{\ell}} = (\ell_1, \ldots, \ell_{d-1})$. We use induction on j to prove that $e_j(\widehat{\boldsymbol{\ell}}) = (-1)^j \cdot \ell_d^j$ for $j = 1, \ldots, d-1$. For $j = 1$, we have

$$0 = e_1(\boldsymbol{\ell}) = (\ell_1 + \cdots + \ell_{d-1}) + \ell_d = e_1(\widehat{\boldsymbol{\ell}}) + \ell_d$$

which proves the statement. For $j = 2, \ldots, d-1$, consider the recursive relation

$$e_j(\boldsymbol{\ell}) = e_j(\widehat{\boldsymbol{\ell}}) + \ell_d e_{j-1}(\widehat{\boldsymbol{\ell}}).$$

By assumption we have $e_j(\boldsymbol{\ell}) = 0$ and the induction hypothesis guarantees $e_{j-1}(\widehat{\boldsymbol{\ell}}) = (-1)^{j-1} \cdot \ell_d^{j-1}$; we deduce $e_j(\widehat{\boldsymbol{\ell}}) = -\ell_d \cdot (-1)^{j-1} \cdot \ell_d^{j-1} = (-1)^j \ell_d^j$ which proves the statement. Finally, notice $f = \alpha e_d(\boldsymbol{\ell}) = \alpha \ell_d \cdot (-1)^{d-1} \cdot e_{d-1}(\widehat{\boldsymbol{\ell}}) = -\alpha \ell_d^d$, which concludes the proof. $\qquad \square$

However, the model is complete if one allows approximations, as shown in [Kum20]. We introduce an equivalence relation on $\mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]$: given two polynomials f_1, f_2 whose coefficients depend rationally on ϵ, we write $f_1 \simeq f_2$ if $\lim_\epsilon f_1, \lim_\epsilon f_2$ are both finite and they coincide. We often use this notation with either f_1 or f_2 not depending on ϵ: if, for instance, f_1 does not depend on ϵ, then $f_1 \simeq f_2$ means that $f_2 = f_1 + O(\epsilon)$.

2.4 Proposition ([Kum20]). *For all homogeneous f we have $\underline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \mathsf{WR}(f)$.*

Proof. The proof is based on a construction by Shpilka [Shp02]. Let $\mathsf{WR}(f) = r$ and write $f = \sum_{i=1}^r \ell_i^d$. Let ζ be a primitive d-th root of unity. Then one verifies that

$$f = -e_d(-\zeta^0 \ell_1, -\zeta^1 \ell_1, \ldots, -\zeta^{d-1} \ell_1, \ldots \ldots, -\zeta^0 \ell_r, -\zeta^1 \ell_r, \ldots, -\zeta^{d-1} \ell_r)$$

and for all $0 < i < d$ we have

$$e_i(-\zeta^0 \ell_1, -\zeta^1 \ell_1, \ldots, -\zeta^{d-1} \ell_1, \ldots \ldots, -\zeta^0 \ell_r, -\zeta^1 \ell_1, \ldots, -\zeta^{d-1} \ell_r) = 0.$$

Hence $f \simeq -\epsilon^{-d}\big(\big((1 - \epsilon \zeta^0 \ell_1) \cdots (1 - \epsilon \zeta^{d-1} \ell_r)\big) - 1\big)$. Therefore $\underline{\mathsf{Kc}}(f) \leq rd$. $\qquad \square$

In fact, the following slightly stronger statement is true:

2.5 Proposition. *For all homogeneous f we have $\underline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \underline{\mathsf{WR}}(f)$.*

Proof. Analogously to the proof in Proposition 2.4, let $\underline{\mathsf{WR}}(f) = r$ and let $\ell_1, \ldots, \ell_r$ be linear forms depending rationally on ϵ such that $f \simeq \sum_{i=1}^r \ell_i^d = -e_d(-\zeta^0 \ell_1, \ldots, -\zeta^{d-1} \ell_r)$. Moreover, for all $0 < i < d$, we have $e_i(-\zeta^0 \ell_1, \ldots, -\zeta^{d-1} \ell_r) = 0$.

Choose M large enough so that for all $d < i \leq dr$ we have that $\epsilon^{-Md} e_i(-\epsilon^M \zeta^0 \ell_1, \ldots, -\epsilon^M \zeta^{d-1} \ell_r) \simeq 0$. We obtain $f \simeq -\epsilon^{-Md}\big(\big((1-\epsilon^M \zeta^0 \ell_1) \cdots (1-\epsilon^M \zeta^{d-1} \ell_r)\big) - 1\big)$. Therefore $\underline{\mathsf{Kc}}(f) \leq rd$. $\qquad \square$

Proposition 2.4 and Proposition 2.5 show that if $\underline{\mathsf{WR}}(f)$ is small then $\underline{\mathsf{Kc}}(f)$ is small. However, there are polynomials with large Waring (border) rank but small Kumar complexity, such as products of linear forms. For instance $\underline{\mathsf{WR}}(x_1 \cdots x_n)$ is exponentially large: a lower bound of $\binom{n}{\lfloor n/2 \rfloor}$ can be easily shown by partial derivative methods, see e.g. [LT10, Sec. 11], [CKW11, Thm. 10.4]. However every completely reducible form has small Kumar complexity:

2.6 Lemma. *If $f = \ell_1 \cdots \ell_d$ is a product of homogeneous linear forms ℓ_i, then $\underline{\mathsf{Kc}}(f) = d$.*

Proof. The lower bound is immediate because $\underline{\mathsf{Kc}}(f) \geq \deg(f)$. For the upper bound, notice $f \simeq \epsilon^d\big(\big(\prod_{i=1}^d (1 + \epsilon^{-1} \ell_i)\big) - 1\big)$. $\qquad \square$

The main result of this section is a converse of the above statements: informally, homogeneous polynomials with small border Waring rank and product of linear forms are the only homogeneous polynomials with small border Kumar complexity. The following result explains the relation between border Waring rank and border Kumar's complexity and completes the proof of Theorem 1.1.

2.7 Theorem. *If f is a product of homogeneous linear forms, then $\underline{\mathsf{Kc}}(f) = \deg(f)$. For all other homogeneous f we have*

$$\max\{\deg(f), \underline{\mathsf{WR}}(f)\} \quad \leq \quad \underline{\mathsf{Kc}}(f) \quad \leq \quad \deg(f) \cdot \underline{\mathsf{WR}}(f).$$

Proof. The first statement is Lemma 2.6. The right inequality follows from Proposition 2.5. Clearly $\deg(f) \leq \underline{\mathsf{Kc}}(f)$. The inequality $\underline{\mathsf{WR}}(f) \leq \underline{\mathsf{Kc}}(f)$ is a combination of Lemma 2.8, Proposition 2.9, and Theorem 2.10 below. $\qquad\square$

Note that in the definition of $\underline{\mathsf{Kc}}$, the factor α can be assumed to be a scalar times a power of ϵ, because only the lowest power of ϵ in α would contribute to the limit. We distinguish three cases, depending on the sign of the exponent of ϵ in α.

- $\underline{\mathsf{Kc}}^+(f)$ is the smallest m such that $f \simeq \gamma\epsilon^N\big(\prod_{i=1}^m(1+\ell_i)-1\big)$ for some $N \geq 1$, $\gamma \in \mathbb{C}$ and $\ell_i \in \mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]_1$; set $\underline{\mathsf{Kc}}^+(f) = \infty$ if such an m does not exist;

- $\underline{\mathsf{Kc}}^-(f)$ is the smallest m such that $f \simeq \gamma\epsilon^{-M}\big(\prod_{i=1}^m(1+\ell_i)-1\big)$ for some $M \geq 1$, $\gamma \in \mathbb{C}$ and $\ell_i \in \mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]_1$; set $\underline{\mathsf{Kc}}^-(f) = \infty$ if such an m does not exist;

- $\underline{\mathsf{Kc}}^=(f)$ is the smallest m such that $f \simeq \gamma\big(\prod_{i=1}^m(1+\ell_i)-1\big)$ for some $\gamma \in \mathbb{C}$ and $\ell_i \in \mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]_1$; set $\underline{\mathsf{Kc}}^=(f) = \infty$ if such an m does not exist.

We observe that $\underline{\mathsf{Kc}}(f) = \min\big\{\underline{\mathsf{Kc}}^+(f),\ \underline{\mathsf{Kc}}^=(f),\ \underline{\mathsf{Kc}}^-(f)\big\}$.

2.8 Lemma. *For all homogeneous f, if $\underline{\mathsf{Kc}}^+(f)$ is finite, then f is a product of homogeneous linear forms.*

Proof. Let $f \simeq \gamma\epsilon^N\big(\prod_{i=1}^m(1+\ell_i)-1\big)$ with $N \geq 1$. Since $\epsilon^N \simeq 0$, we have $f \simeq \gamma\epsilon^N\prod_{i=1}^m(1+\ell_i)$, namely f is limit of a product of affine linear polynomials. The property of being completely reducible is closed, therefore we deduce that f is a product of affine linear polynomials. Since f is homogeneous, its factors are homogeneous as well. $\qquad\square$

2.9 Proposition (Newton Identities). *For all homogeneous f we have $\mathsf{WR}(f) \leq \mathsf{Kc}(f) = \underline{\mathsf{Kc}}^=(f)$.*

Proof. Let $d := \deg(f)$. Suppose $\underline{\mathsf{Kc}}^=(f) = m$ and write $f \simeq f_\epsilon := \gamma\big(\prod_{i=1}^m(1+\ell_i)-1\big)$. One can verify that if even one of the ℓ_i diverges, then the j-th homogeneous part of f_ϵ diverges, where j is the number of diverging ℓ_i. Hence all ℓ_i converge and we can set ϵ to zero. Hence, $\underline{\mathsf{Kc}}^=(f) = \mathsf{Kc}(f)$. Now, since f is homogeneous, each homogeneous degree i part of f_ϵ vanishes, $i < d$. In other words, $e_i(\boldsymbol{\ell}) = 0$ for all $1 \leq i < d$, where $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m)$. Hence $s(\boldsymbol{\ell}) = 0$ for *all* symmetric polynomials of degree $< d$. Therefore the Newton identity $p_d = (-1)^{d-1} \cdot d \cdot e_d + \sum_{i=1}^{d-1}(-1)^{d+i-1}e_{d-i} \cdot p_i$ gives that $e_d(\boldsymbol{\ell})$ and $p_d(\boldsymbol{\ell})$ are same up to multiplication by a scalar. Hence $\mathsf{WR}(f) \leq m$. $\qquad\square$

2.10 Theorem (Border Newton Identities). *For all homogeneous f: $\underline{\mathsf{WR}}(f) \leq \underline{\mathsf{Kc}}^-(f)$.*

Proof. Let $d := \deg(f)$. Let $f \simeq f_\epsilon := \gamma\epsilon^{-M}\big(\prod_{i=1}^m(1+\ell'_i)-1\big)$ with $M \geq 1$. From the convergence of f_ϵ we deduce that for each i we have $\ell'_i = \epsilon\ell_i$ with $\ell_i \in \mathbb{C}[\epsilon][\boldsymbol{x}]_1$, because otherwise the homogeneous degree j part diverges, where j is the number of ℓ'_i that do not satisfy this property.

Now, let $f_{\epsilon,j}$ denote the homogeneous degree j part of f_ϵ. Since f is homogeneous of degree d, for $0 \leq j < d$ we have $f_{\epsilon,j} \simeq 0$. By expanding the product, observe that for all $0 < j < d$ we have $0 \simeq f_{\epsilon,j} = \gamma\epsilon^{-M}e_j(\epsilon\ell_1, \ldots, \epsilon\ell_m) = \gamma\epsilon^{-M+j}e_j(\ell_1, \ldots, \ell_m)$. We now show by induction that for all $1 \leq j < d$ we have $\epsilon^{-M+j}p_j(\ell_1, \ldots, \ell_m) \simeq 0$. This is clear for $j = 1$, because $p_1 = e_1$. For the step from j to $j+1$ we use Newton's identities:

$$p_{j+1} = (-1)^j (j+1) e_{j+1} + \sum_{i=1}^j (-1)^{j+i} e_{j+1-i} \cdot p_i.$$

Hence $\epsilon^{-M+(j+1)}p_{j+1}(\boldsymbol{\ell})$

$$= (-1)^j (j+1) \underbrace{\epsilon^{-M+(j+1)}e_{j+1}(\boldsymbol{\ell})}_{\simeq 0} + \sum_{i=1}^j (-1)^{j+i} \underbrace{\epsilon^{-M+(j+1)-i}e_{j+1-i}(\boldsymbol{\ell})}_{\simeq 0} \cdot \underbrace{\epsilon^M}_{\simeq 0} \cdot \underbrace{\epsilon^{-M+i}p_i(\boldsymbol{\ell})}_{\simeq 0} \simeq 0.$$

This finishes the induction proof, now we use Newton's identities again in the same way to see that $\epsilon^{-M+d}p_d(\boldsymbol{\ell}) \simeq (-1)^{d-1} \cdot d \cdot \epsilon^{-M+d}e_d(\boldsymbol{\ell})$:

$$\epsilon^{-M+d}p_d(\boldsymbol{\ell}) = (-1)^{d-1} \cdot d \cdot \epsilon^{-M+d}e_d(\boldsymbol{\ell}) + \sum_{i=1}^{d-1}(-1)^{d-1+i}\underbrace{\epsilon^{-M+d-i}e_{d-i}(\boldsymbol{\ell})}_{\simeq 0} \cdot \underbrace{\epsilon^M}_{\simeq 0} \cdot \underbrace{\epsilon^{-M+i}p_i(\boldsymbol{\ell})}_{\simeq 0}.$$

We are done now, because $f \simeq f_{\epsilon,d} = \gamma\epsilon^{-M+d}e_d(\ell_1,\ldots,\ell_m) \simeq \gamma\epsilon^{-M+d} \cdot \frac{1}{d} \cdot (-1)^{d-1}p_d(\ell_1,\ldots,\ell_m)$ and hence $\underline{\mathsf{WR}}(f) \leq m$. $\qquad\square$

2.1 Linear approximations and Waring rank

We demonstrated the inequality $\underline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \mathsf{WR}(f)$ in Proposition 2.4. In the proof of Proposition 2.4, only "linear approximations" have been used; we prove here a converse of Proposition 2.4 in the restricted setting of linear approximation. Given a homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]_d$, let $\underline{\mathsf{Kc}_1^-}(f)$ be the smallest m such that there exist linear forms $\ell_1,\ldots,\ell_m \in \mathbb{C}[\mathbf{x}]_1$ and $M \geq 1$ such that $f \simeq \gamma\epsilon^{-M}\big(\prod_{i=1}^m(1+\epsilon\ell_i)-1\big)$.

2.11 Proposition. *For any homogeneous polynomial f of degree d, we have* $\mathsf{WR}(f) \leq \underline{\mathsf{Kc}_1^-}(f) \leq d \cdot \mathsf{WR}(f)$.

Proof. The inequality $\underline{\mathsf{Kc}_1^-}(f) \leq d \cdot \mathsf{WR}(f)$ is clear from the proof of Proposition 2.4, as there we obtained an expression of the form described in the definition of $\underline{\mathsf{Kc}_1^-}$. Suppose $\underline{\mathsf{Kc}_1^-}(f) = m$ and write $f \simeq f_\epsilon := \gamma\epsilon^{-M}\big(\prod_{i=1}^m(1+\epsilon\ell_i)-1\big)$ with $M \geq 1$ and $\ell_i \in \mathbb{C}[\boldsymbol{x}]_1$. It is immediate that $m \geq M$, $f = \gamma e_M(\boldsymbol{\ell})$ and $e_j(\boldsymbol{\ell}) = 0$ for $j < M$, where $\boldsymbol{\ell} = (\ell_1,\ldots,\ell_m)$. Via the Newton identity for the power sum polynomial, we have

$$p_M(\boldsymbol{\ell}) = (-1)^{M-1}Me_M(\boldsymbol{\ell}) + \sum_{i=1}^{M-1}(-1)^{M+i-1}e_{M-i}(\boldsymbol{\ell}) \cdot p_i(\boldsymbol{\ell}).$$

Since $e_j(\boldsymbol{\ell}) = 0$ for all $1 \leq j < M$, we obtain:

$$p_M(\boldsymbol{\ell}) = (-1)^{M-1}Me_M(\boldsymbol{\ell}) = \tfrac{1}{\gamma}(-1)^{M-1}Mf.$$

We conclude $\mathsf{WR}(f) = \mathsf{WR}(p_M(\boldsymbol{\ell})) \leq \mathsf{WR}(p_M) = m = \underline{\mathsf{Kc}_1^-}(f)$, as desired. $\qquad\square$

3 Restricted binomials: debordering and lower bounds

In this section, we study restricted binomials. A binomial bn_d is the polynomial $\mathrm{bn}_d(\boldsymbol{x},\boldsymbol{y}) := P_{2,0}^{[d]} = x_1\ldots x_d + y_1\cdots y_d$. Theorem 1.3 is based on the presentation of $P_{1,1}^{[d]}$ and $P_{1,2}^{[d]}$ as restrictions of the binomial $P_{2,0}^{[d]}$, which follows from the fact that both x_0^d and $x_0^d - x_{d+1}^d = \prod_{i=1}^d(x_0 - \zeta^i x_{d+1})$ are completely reducible; here ζ is a primitive d-th root of 1. Therefore, degenerations of $P_{1,1}^{[d]}$ and $P_{1,2}^{[d]}$ arise as limits of the sum of two products

$$\lim_{\epsilon\to 0}\left(\prod_{i=1}^d \ell_i(\epsilon) + \prod_{i=1}^d \ell_i'(\epsilon)\right)$$

where $\ell_i(\epsilon), \ell_i'(\epsilon)$ are linear forms depending rationally on ϵ, and the second product is *restricted*, in the sense that, up to change of coordinates, it has either one or two variables.

In Section 3.1, we deborder product-plus-power $(P_{1,1}^{[d]})$ and product-plus-two-powers models $(P_{1,2}^{[d]})$. In Section 3.2, we show exponential gaps between product-plus-power, product-plus-two-powers, and binomials (in the affine sense). Identifying explicit polynomials which are hard to approximate, and proving it remains a major template in algebraic and geometric complexity theory. Often, proving lower bounds on the homogeneous model turns out to be easier than in its affine model, because of the non-trivial cancellations in the latter model. However, in the restricted setting, we are able to show *optimal* lower bounds, see Theorem 3.15 & Theorem 3.16.

3.1 Debordering: Characterizing special binomials

In this section we prove debordering results for product-plus-power and product-plus-two-powers models. Our method applies also for more general computational model based on restricted binomials. More specifically, we prove that polynomials obtained in the limit in our model have low border Waring rank. One can then apply a debordering result for $\underline{\mathsf{WR}}$ such as $\mathsf{abpw}(f) \leq \underline{\mathsf{WR}}(f)$ [BDI21, For14] or the results from [DGI$^+$24] to get a complete debordering.

3.1 Definition (Restricted binomial model)**.** *We say that a homogeneous degree d polynomial f is in the* class RB_k *if it can be presented as*

$$f = \prod_{i=1}^{d} \ell_i + \prod_{i=1}^{d} \ell_i'$$

for some linear forms ℓ_i, ℓ_i' such that $\mathrm{rank}(\ell_1', \ldots, \ell_d') \leq k$. We also define the corresponding approximate class $\overline{RB}_k$ in the standard way: a homogeneous degree d polynomial f is in $\overline{RB}_k$ if

$$f = \lim_{\varepsilon \to 0} \left(\prod_{i=1}^{d} \ell_i(\epsilon) + \prod_{i=1}^{d} \ell_i'(\epsilon) \right) \tag{3.2}$$

for some $\ell_i(\epsilon), \ell_i'(\epsilon) \in \mathbb{C}[\epsilon^{\pm 1}][\mathbf{x}]_1$ such that $\mathrm{rank}(\ell_1'(\epsilon), \ldots, \ell_d'(\epsilon)) \leq k$ for every $\epsilon \neq 0$.

The main theorem of this section is a debordering result of $\overline{RB}_k$ in terms of border Waring rank. Theorem 1.3 is a consequence of this result.

3.3 Theorem (Debordering $\overline{RB}_k$)**.** *Let f be a homogeneous polynomial of degree d in $\overline{RB}_k$. Then either $f \in RB_k$, or $\underline{\mathsf{WR}}(f) \leq O(d^{3k+2})$.*

To prove this theorem, we first need some basic lemmas which will be used in the proof. We will use non-homogeneous polynomials, so instead of Waring rank we will be working with the complexity of $\Sigma\Lambda\Sigma$-circuits. Denote by $\Sigma^{[s]}\Lambda^{[e]}\Sigma$ the class of (non-homogeneous) polynomials representable as a sum of s powers of affine linear forms with exponents not exceeding e, and by $\overline{\Sigma^{[s]}\Lambda^{[e]}\Sigma}$ the corresponding class closed under approximation. As the following lemma shows, for homogeneous polynomials this model is equal in power to border Waring rank.

3.4 Lemma. *Let f be a homogeneous polynomial of degree d. Then $f \in \overline{\Sigma^{[s]}\Lambda\Sigma}$ if and only if $\underline{\mathsf{WR}}(f) \leq s$*

Proof. Clearly, if $\underline{\mathsf{WR}}(f) \leq s$ then $f \in \overline{\Sigma^{[s]}\Lambda\Sigma}$. For the converse, suppose $f \simeq \sum_{i \in [s]} (\alpha_i + \ell_i)^{e_i}$, where $\alpha_i \in \mathbb{C}[\epsilon^{\pm 1}]$, and $\ell_i \in \mathbb{C}[\epsilon^{\pm 1}][\boldsymbol{x}]_1$. Taking the degree d part of each side, we obtain a border Waring rank decomposition $f \simeq \sum_{i\,:\,e_i \geq d} \binom{e_i}{d} \ell_i^d \alpha_i^{e_i - d}$ with at most s summands. $\qquad\square$

We recall a classical result on the border Waring rank of a binary monomial.

3.5 Proposition (see, e.g., [LT10, Cor. 4.5])**.** *If $a \leq b$, then $\underline{\mathsf{WR}}(x^a y^b) = a + 1$.*

The next lemma bounds the $\Sigma\Lambda\Sigma$ complexity of a polynomial in terms of the complexity of polynomials obtained from it by substitution of variables.

3.6 Lemma (Interpolation)**.** *Let f be a polynomial of degree d such that $f(\gamma_i, x_2, \ldots, x_n) \in \overline{\Sigma^{[s]} \wedge^{[e]} \Sigma}$ for some distinct $\gamma_0, \ldots, \gamma_d \in \mathbb{C}$. Then $f \in \overline{\Sigma^{[s(d+1)^3] \wedge^{[e+d]} \Sigma}}$.*

Proof. Write $f(\boldsymbol{x}) = \sum_{j=0}^{d} x_1^j f_j(x_2, \ldots, x_n)$. By polynomial interpolation there exist $\alpha_{ij} \in \mathbb{C}$ such that $f_j = \sum_{i=0}^{d} \alpha_{ij} f(\gamma_i, x_2, \ldots, x_n)$. By assumption, $f(\gamma_i, x_2, \ldots, x_n) \simeq \sum_{j=1}^{s} \ell_{ij}^{e_j}$, where ℓ_{ij} are affine linear forms with coefficients in $\mathbb{C}[\epsilon^{\pm 1}]$, and $e_j \leq e$. Hence

$$f_j(\boldsymbol{x}) \simeq \sum_{i=0}^{d} \sum_{j=1}^{s} \alpha_{ij} \ell_{ij}^{e_j} \implies f_j(\boldsymbol{x}) \in \overline{\Sigma^{[s(d+1)] \wedge^{[e]} \Sigma}} \,.$$

Note that for any affine linear polynomial ℓ the polynomial $x_1^j \ell^e$ can be approximated by a $\Sigma^{[d+1]} \wedge^{[e+j]} \Sigma$-circuit using the decomposition of the monomial $x^j y^e$ with border Waring rank equal to $\min\{j+1, e+1\} \leq j+1 \leq d+1$; this follows from Proposition 3.5. Therefore $x_1^j f_j \in \overline{\Sigma^{[s(d+1)^2]} \wedge^{[e+j]} \Sigma}$, and $f(\boldsymbol{x}) = \sum_{i=0}^d x_1^j f_j \in \overline{\Sigma^{[s(d+1)^3]} \wedge^{[e+d]} \Sigma}$. $\qquad\square$

Applying Lemma 3.6 several times we obtain the following result.

3.7 Corollary. *Let $f(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$ be a polynomial of degree d such that*

$$f(\gamma_{1i_1}, \gamma_{2i_2}, \ldots, \gamma_{ki_k}, x_{k+1}, \ldots, x_n) \in \overline{\Sigma^{[s]} \wedge^{[e]} \Sigma}$$

for some $\gamma_{ij} \in \mathbb{C}$, $1 \leq i \leq k$, $0 \leq j \leq d$, with $\gamma_{i0}, \ldots, \gamma_{id}$ distinct for each i. Then $f \in \overline{\Sigma^{[s(d+1)^{3k}]} \wedge^{[e+kd]} \Sigma}$.

Additionally, we need the following statement similar to Theorem 2.10, which considers an auxiliary Kumar-like model.

3.8 Theorem. *For any degree d polynomial $f(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$, not necessarily homogeneous, suppose we have $f \simeq \epsilon^{-M}\left(\prod_{i=1}^m (1+\epsilon a_i) - \prod_{i=1}^m (1+\epsilon b_i)\right)$ for some linear forms $a_i, b_i \in \mathbb{C}[\epsilon][\boldsymbol{x}]_1$ with $M \geq 1$. Then $f \in \overline{\Sigma^{[2md]} \wedge^{[d]} \Sigma}$.*

Proof. Let $f_\epsilon = \epsilon^{-M}\left(\prod_{i=1}^m (1+\epsilon a_i) - \prod_{i=1}^m (1+\epsilon b_i)\right)$. Denote by f_j and $f_{\epsilon,j}$ the homogeneous degree j parts of f and f_ϵ respectively. Since $f \simeq f_\epsilon$, we have

$$f_j \simeq f_{\epsilon,j} = \epsilon^{-M}\left(e_j(\epsilon a_1, \ldots, \epsilon a_m) - e_j(\epsilon b_1, \ldots, \epsilon b_m)\right) = \epsilon^{-M+j}\left(e_j(\boldsymbol{a}) - e_j(\boldsymbol{b})\right),$$

where $\boldsymbol{a} = (a_1, \ldots, a_m)$ and similarly $\boldsymbol{b} = (b_1, \ldots, b_m)$. Note that since $f_{\epsilon,j}$ converges, $e_j(\boldsymbol{a}) - e_j(\boldsymbol{b})$ is divisible by ϵ^{M-j} for all $j \geq 1$, that is,

$$e_j(\boldsymbol{a}) \equiv e_j(\boldsymbol{b}) \bmod \langle \epsilon^{M-j} \rangle$$

where we consider $e_j(\boldsymbol{a})$ and $e_j(\boldsymbol{b})$ as elements of the ring $\mathbb{C}[\epsilon][\mathbf{x}]$.

We now show by induction that for all $j \geq 1$ the following additional congruences hold.

$$p_j(\boldsymbol{a}) \equiv p_j(\boldsymbol{b}) \bmod \langle \epsilon^{M-j} \rangle$$
$$p_j(\boldsymbol{a}) - p_j(\boldsymbol{b}) \equiv (-1)^{j-1} j \left(e_j(\boldsymbol{a}) - e_j(\boldsymbol{b})\right) \bmod \langle \epsilon^{M-j+1} \rangle$$

The case $j = 1$ is trivially true because $p_1 = e_1$. For the induction step from j to $j+1$, we use Newton's identities

$$p_{j+1} = (-1)^j (j+1) e_{j+1} + \sum_{i=1}^j (-1)^{j+i} e_{j+1-i} \cdot p_i.$$

We obtain

$$p_{j+1}(\boldsymbol{a}) - p_{j+1}(\boldsymbol{b}) = (-1)^j (j+1)\left(e_{j+1}(\boldsymbol{a}) - e_{j+1}(\boldsymbol{b})\right)$$
$$+ \sum_{i=1}^j (-1)^{j+i}\left(e_{j+1-i}(\boldsymbol{a}) \cdot p_i(\boldsymbol{a}) - e_{j+1-i}(\boldsymbol{b}) \cdot p_i(\boldsymbol{b})\right). \qquad (3.9)$$

By induction hypothesis we know that for $1 \leq i \leq j$

$$p_i(\boldsymbol{a}) \equiv p_i(\boldsymbol{b}) \bmod \langle \epsilon^{M-i} \rangle$$
$$e_{j+1-i}(\boldsymbol{a}) \equiv e_{j+1-i}(\boldsymbol{b}) \bmod \langle \epsilon^{M-(j+1)+i} \rangle.$$

Since $M - j \leq M - i$ and $M - j \leq M - (j+1) + i$, this can be relaxed to

$$p_i(\boldsymbol{a}) \equiv p_i(\boldsymbol{b}) \bmod \langle \epsilon^{M-j} \rangle$$
$$e_{j+1-i}(\boldsymbol{a}) \equiv e_{j+1-i}(\boldsymbol{b}) \bmod \langle \epsilon^{M-j} \rangle.$$

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From (3.9) we get

$$p_{j+1}(\boldsymbol{a}) - p_{j+1}(\boldsymbol{b}) \equiv (-1)^j (j+1) \left(e_{j+1}(\boldsymbol{a}) - e_{j+1}(\boldsymbol{b}) \right) \bmod \langle \epsilon^{M-j} \rangle.$$

Weakening this to an equivalence $\bmod \langle \epsilon^{M-(j+1)} \rangle$, we obtain

$$p_{j+1}(\boldsymbol{a}) - p_{j+1}(\boldsymbol{b}) \equiv (-1)^j (j+1) \left(e_{j+1}(\boldsymbol{a}) - e_{j+1}(\boldsymbol{b}) \right) \equiv 0 \bmod \langle \epsilon^{M-(j+1)} \rangle,$$

or $p_{j+1}(\boldsymbol{a}) \equiv p_{j+1}(\boldsymbol{b}) \bmod \langle \epsilon^{M-(j+1)} \rangle$, finishing the induction.

Finally, we use the proved congruences to write an approximate decomposition of f. We have

$$f_j \simeq \epsilon^{-M+j} \left(e_j(\boldsymbol{a}) - e_j(\boldsymbol{b}) \right) \simeq \epsilon^{-M+j} \cdot \frac{1}{j} \cdot (-1)^{j-1} \left(p_j(\boldsymbol{a}) - p_j(\boldsymbol{b}) \right),$$

which shows that $\underline{\mathsf{WR}}(f_j) \leq 2m$. Note that $f_0 = 0$, so $f = \sum_{j=1}^d f_j \in \overline{\Sigma^{[2md]} \Lambda^{[d]} \Sigma}$. □

3.10 Corollary. *For any degree d polynomial $f(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$, not necessarily homogeneous, suppose we have $f \simeq \epsilon^{-M} \big(\alpha \prod_{i=1}^m (1 + \epsilon a_i) - \beta \prod_{i=1}^m (1 + \epsilon b_i) \big)$ with $M \geq 1$ for some $a_i, b_i \in \mathbb{C}[\epsilon][\boldsymbol{x}]_1$ and $\alpha, \beta \in \mathbb{C}[\epsilon]$ such that $\alpha \simeq \beta \not\simeq 0$. Then $f \in \overline{\Sigma^{[2md+1]} \wedge^{[d]} \Sigma}$.*

Proof. Let f_j and $f_{\epsilon,j}$ be the homogeneous parts as in the proof of the preceding Theorem. Additionally, Let $\alpha_0 = \lim_{\epsilon \to 0} \alpha$ and $\gamma' = \frac{\beta}{\alpha} \in \mathbb{C}[[\epsilon]]$. As mentioned earlier, one can truncate and work with $\gamma \equiv \gamma' \bmod \langle \epsilon^r \rangle$, for some large positive integer r. From assumptions of the theorem, $\alpha_0 \neq 0$ and $\gamma \simeq 1$. We have

$$\frac{1}{\alpha_0} f \simeq \frac{1}{\alpha} f \simeq \epsilon^{-M} \big(\prod_{i=1}^m (1 + \epsilon a_i) - \gamma \prod_{i=1}^m (1 + \epsilon b_i) \big)$$

By taking degree 0 part we get $\frac{1}{\alpha_0} f_0 \simeq \frac{1}{\alpha_0} f_{\epsilon,0} = \epsilon^{-M}(1 - \gamma)$, so for $j \geq 1$ we have

$$\frac{1}{\alpha_0} f_j \simeq \epsilon^{-M+j} (e_j(\boldsymbol{a}) - \gamma e_j(\boldsymbol{b})) = \epsilon^{-M+j}(e_j(\boldsymbol{a}) - e_j(\boldsymbol{b})) + \epsilon^j \frac{f_{\epsilon,0}}{\alpha_0} e_j(\boldsymbol{b}) \simeq \epsilon^{-M+j}(e_j(\boldsymbol{a}) - e_j(\boldsymbol{b})),$$

hence

$$f \simeq f_0 + \alpha_0 \epsilon^{-M} \big(\prod_{i=1}^m (1 + \epsilon a_i) - \prod_{i=1}^m (1 + \epsilon b_i) \big),$$

and we reduce to the case considered in Theorem 3.8. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Since $f \in \overline{RB}_k$, it has an approximate decomposition (3.2), which we rewrite as

$$f \simeq \epsilon^p \prod_{i=1}^d \ell_i - \epsilon^{p'} \prod_{i=1}^d \ell'_i$$

where $\ell_i, \ell'_i \in \mathbb{C}[\epsilon][\mathbf{x}]_1$ are not divisible by ϵ and $\mathrm{rank}(\ell'_1, \ldots \ell'_d) \leq k$ at any $\epsilon \neq 0$. Define $\ell_{i0} \in V$ as $\ell_{i0} = \ell_i|_{\epsilon=0}$ and similarly $\ell'_{i0} = \ell'_i|_{\epsilon=0}$. ℓ_{i0} and ℓ'_{i0} are nonzero and by semicontinuity of rank we have $\mathrm{rank}(\ell'_{10}, \ldots \ell'_{d0}) \leq k$.

If $p = p' = 0$, then $f = \prod_{i=0}^d \ell_{i0} - \prod_{i=0}^d \ell'_{i0}$. Similarly, if one of the exponents p and p' is positive, then the corresponding summand tends to 0 as $\epsilon \to 0$, and f is a product of linear forms, and if both p and p' are positive, then $f = 0$. In all these cases we have $f \in RB_k$.

Consider now the case when there are negative exponents. The convergence of the right hand side of the decomposition implies that $p = p'$ and the lowest degree term $\prod_{i=0}^d \ell_{i0} - \prod_{i=0}^d \ell'_{i0}$ is zero. By unique factorization the sets of linear forms ℓ_{i0} and ℓ'_{i0} are the same up to scalar multiples, and we can permute and rescale the factors in one of the products so that $\ell_{i0} = \ell'_{i0}$. Additionally we can assume that $\ell_{10}, \ldots, \ell_{r0}$ are linearly independent, where $r = \mathrm{rank}(\ell_{10}, \ldots \ell_{d0}) \leq k$.

Since ℓ_{i0} for $i \leq r$ are linearly independent, there exists an invertible linear map A such that $\ell_{i0}(A\mathbf{x}) = x_i$ for $i \leq r$. The linear forms ℓ_{i0} lie in the linear span of the first r of them, which means that $\ell_{i0}(A\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_r]_1$ for all i.

Let $M = -p$, $L_i(\mathbf{x}) = \ell_i(A\mathbf{x})$ and $L_i'(\mathbf{x}) = \ell_i'(A\mathbf{x})$. For the polynomial $g(\mathbf{x}) = f(A\mathbf{x})$ we obtain an approximate decomposition of the following form

$$g \simeq \epsilon^{-M}\big(\prod_{i=1}^{d} L_i - \prod_{i=1}^{d} L_i'\big)$$

where $L_i, L_i' \in \mathbb{C}[\epsilon][\mathbf{x}]_1$ are such that $L_{i0} := L_i|_{\epsilon=0} = L_i'|_{\epsilon=0}$ are nonzero elements of $\mathbb{C}[x_1, \ldots, x_r]$.

Choose $\gamma_{ij} \in \mathbb{C}$ for $1 \leq i \leq r$, $0 \leq j \leq d$ so that $\gamma_{i0}, \ldots, \gamma_{id}$ are distinct for each i and $L_{i0}(\gamma_{1j_1}, \ldots, \gamma_{rj_r}) \neq 0$ for all $i, j_1, \ldots, j_r$. The choice is possible because L_{k0} are nonzero and hence the set of tuples γ not satisfying the required conditions is a nontrivial Zariski closed set. Write

$$L_i(\gamma_{1j_1}, \ldots, \gamma_{rj_r}, x_{r+1}, \ldots, x_n) = \alpha_i + \epsilon A_i(x_{r+1}, \ldots, x_n)$$
$$L_i'(\gamma_{1j_1}, \ldots, \gamma_{rj_r}, x_{r+1}, \ldots, x_n) = \beta_i + \epsilon B_i(x_{r+1}, \ldots, x_n)$$

with $\alpha_i, \beta_i \in \mathbb{C}[\epsilon]$ such that $\alpha_i \simeq \beta_i$ and $A_i, B_i \in \mathbb{C}[\epsilon][x_{r+1}, \ldots, x_n]_1$. Set $\alpha = \prod_{i=1}^{d} \alpha_i$, $\beta = \prod_{i=1}^{d} \beta_i$, $a_i = \frac{A_i}{\alpha_i}$, $b_i = \frac{B_i}{\beta_i}$. Because $\alpha_i|_{\epsilon=0} = L_{i0}(\gamma_{1j_1}, \ldots, \gamma_{rj_r}) \neq 0$, a_i are well defined in the ring $\mathbb{C}[[\epsilon]][\mathbf{x}]$; ditto for b_i. As argued earlier, truncating and working with finite precision of ϵ suffices, therefore, let $a_i' := a_i \bmod \langle \epsilon^r \rangle$, and similarly for b_i', for some large positive integer r. We obtain

$$g(\gamma_{1j_1}, \ldots, \gamma_{rj_r}, x_{r+1}, \ldots, x_n) \simeq \epsilon^{-M}\big(\alpha \prod_{i=1}^{d}(1 + \epsilon a_i') - \beta \prod_{i=1}^{d}(1 + \epsilon b_i')\big).$$

By Corollary 3.10 $g(\gamma_{1j_1}, \ldots, \gamma_{rj_r}, \mathbf{x}) \in \overline{\Sigma^{[2d^2+1]}\Lambda^{[d]}\Sigma}$. By Lemma 3.6 $g \in \overline{\Sigma^{[(2d^2+1)(d+1)^{3r}]}\Lambda^{[(r+1)d]}\Sigma}$, and by Lemma 3.4 $\underline{\mathsf{WR}}(g) \leq (2d^2+1)(d+1)^{3r} = O(d^{3k+2})$. Since border Waring rank is invariant under invertible linear transformations, the same is true for f. $\qquad \square$

As special cases we obtain the following results for product-plus-power and product-plus-two powers. Note that RB_1 consists of polynomials of the form $\prod_{i=1}^{d} \ell_i + \ell_1'^d$, which are exactly the restrictions of $P_{1,1}^{[d]}$. Similarly, $f \in \overline{RB}_1$ if and only if $f \trianglelefteq P_{1,1}^{[d]}$. Therefore, Theorem 1.3 is a consequence of Theorem 3.3.

Similarly, the result for the product-plus-two-powers follows for the analysis of $\overline{RB}_2$, because the sum of two powers $x_0^d - x_{d+1}^d$ can be represented as a product of linear forms in two variables $x_0^d - x_{d+1}^d = \prod_{i=1}^{d}(x_0 - \zeta^i x_{d+1})$, where ζ is a primitive d-th root of unity. A more careful case-by-case analysis gives the following result.

3.11 Theorem (debordering product-plus-two-powers)**.** *Let $f \in \mathbb{C}[x_1, \cdots, x_n]_d$ such that $f \trianglelefteq P_{1,2}^{[d]}$. One of the three alternatives is true:*

1. *$f \leq P_{1,2}^{[d]}$, or*

2. *$f \leq \prod_{i=1}^{d} y_i + y_0^{d-1} \cdot y_{d+1}$, or*

3. *$\underline{\mathsf{WR}}(f) = O(d^8)$.*

Proof. The proof mostly follows the proof of Theorem 3.3. For the completeness, we give a detailed proof.

Since, $f \trianglelefteq P_{1,2}^{[d]}$, by definition, $f = \lim_{\varepsilon \to 0}(A + B + C)$, where $A := \prod_{i=1}^{d} \ell_i(\varepsilon)$, $B := \ell'(\varepsilon)^d$, and $C := \ell''(\varepsilon)^d$. There are a few cases to analyze.

(I) If individually, $\lim_{\varepsilon \to 0} A$, $\lim_{\varepsilon \to 0} B$, and $\lim_{\varepsilon \to 0} C$ exist, then $f \leq P_{1,2}^{[d]}$.

(II) If $g := \lim_{\varepsilon \to 0}(A + B)$ and $h := \lim_{\varepsilon \to 0} C$ exist, then note that $g \trianglelefteq P_{1,1}^{[d]}$, then by Theorem 1.3, we have

    (i) either $g \leq P_{1,1}^{[d]}$

    (ii) or $\underline{\mathsf{WR}}(g) \leq O(d^5)$.

Since, $\underline{\mathsf{WR}}(h) = 1$, and $f = g + h$, the theorem follows.

(III) If $g := \lim_{\varepsilon \to 0} A$ and $h := \lim_{\varepsilon \to 0}(B + C)$ exist, then note that $g \leq P_{1,0}^{[d]}$, and $\underline{\mathsf{WR}}(h) = 2$. It is known that this either $\mathsf{WR}(h) = 2$ or $h = \widehat{\ell}_1^{d-1}\widehat{\ell}_2$ for two linear forms $\widehat{\ell}_1, \widehat{\ell}_2$, see, e.g., [LT10]. Therefore, $f = g + h$ corresponds to either (1) or (2).

(IV) If none of (I)–(III) is true, then note that f can be rewritten as

$$\lim_{\varepsilon \to 0}\left(\prod_{i=1}^{d} \ell_i(\varepsilon) + \prod_{i=1}^{d}(\ell'(\varepsilon) - \zeta^{2i-1}\ell''(\varepsilon))\right),$$

where ζ is the $(2d)$-th primitive root of unity. Further, it is easy to see that $\mathrm{rank}\left(\ell'(\varepsilon) - \zeta\ell''(\varepsilon), \cdots, \ell'(\varepsilon) - \zeta^{2d-1}\ell''(\varepsilon)\right) \leq 2$, for every $\varepsilon \neq 0$. Therefore, by definition $f \in \overline{RB}_2$. Using, Theorem 3.3, we get that $\underline{\mathsf{WR}}(f) \leq O(d^8)$.

This finishes the proof. $\qquad\square$

3.2 Lower Bounds

In this section, we prove several exponential separations between related polynomials contained in the affine closure of binomials.

3.12 Lemma. *The polynomial* $P_{1,2}^{[d]} = \prod_{i \in [d]} x_i + x_{d+1}^d + x_{d+2}^d$ *cannot be written as a product of linear forms.*

Proof. For every homogeneous polynomial f of degree d which is a product of linear forms, the space of first order partial derivatives has dimension at most d. But $\prod_{i \in [d]} x_i + x_{d+1}^d + x_{d+2}^d$ clearly has $d + 2$ linearly independent partial derivatives. $\qquad\square$

3.13 Lemma. *The polynomial* $P_{2,0}^{[d]} = \prod_{i=1}^{d} x_i + \prod_{i=d+1}^{2d} x_i$ *cannot be written as a product of linear forms.*

Proof. It easily follows from a proof similar to that of Lemma 3.12. $\qquad\square$

3.14 Lemma. *For the polynomial* $P_{1,2}^{[d]} = \prod_{i \in [d]} x_i + x_{d+1}^d + x_{d+2}^d$, *we have* $\underline{\mathsf{WR}}(f) \geq 2^{\Omega(d)}$.

Proof. Evaluating $x_{d+1} = x_{d+2} = 0$, we obtain

$$\underline{\mathsf{WR}}(f) \geq \underline{\mathsf{WR}}(x_1 \cdots x_d) \geq \binom{d}{\lceil d/2 \rceil}$$

where the second inequality follows computing the dimension of the space of partial derivatives of order $\lfloor d/2 \rfloor$, see, e.g., [LT10, Prop. 11.6]. $\qquad\square$

For polynomials f and g (not necessarily homogeneous) we write $f \leq_{\mathrm{aff}} g$ if there exists an affine linear map A with $f = g \circ A$. We write $f \trianglelefteq_{\mathrm{aff}} g$ if there exist affine linear maps A_ϵ with $f = \lim_{\epsilon \to 0} g \circ A_\epsilon$.

3.15 Theorem (First exp. gap theorem)**.** *If* $P_{1,2}^{[d]} \trianglelefteq_{\mathrm{aff}} P_{1,1}^{[e]}$, *then* $e \geq \exp(d)$.

We remark that by Kumar's result [Kum20], we know that there exists $e \leq \exp(d)$, such that $P_{1,2}^{[d]} \trianglelefteq_{\mathrm{aff}} P_{1,1}^{[e]}$. Therefore, Theorem 3.15 is *optimal*.

Proof of Theorem 3.15. Let $P_{1,2}^{[d]} \trianglelefteq_{\text{aff}} P_{1,1}^{[e]}$. That means that there are affine linear forms $L_i \in \mathbb{C}[\epsilon^{\pm 1}][\boldsymbol{x}]$ such that $\prod_{i \in [d]} x_i + x_{d+1}^d + x_{d+2}^d + \epsilon \cdot S(\boldsymbol{x}, \epsilon) = \prod_{i \in [e]} L_i + L_{e+1}^e$. By substituting, $x_i \mapsto x_i / x_0$, and multiplying both sides by x_0^e, we get that $x_0^{e-d} \cdot P_{1,2}^{[d]} + \epsilon \cdot \widehat{S} = \prod_{i \in [e]} \widehat{L}_i + \widehat{L}_{e+1}^e$, for homogeneous linear forms $\widehat{L}_i$, or, equivalently, $x_0^{e-d} \cdot P_{1,2}^{[d]} \trianglelefteq P_{1,1}^{[e]}$.

By Theorem 1.3, we know that $x_0^{e-d} \cdot P_{1,2}^{[d]} \trianglelefteq P_{1,1}^{[e]}$ implies either (i) $x_0^{e-d} \cdot P_{1,2}^{[d]} = \prod_{i \in [e]} \ell_i + \ell_0^e$, for some linear forms $\ell_i \in \mathbb{C}[\boldsymbol{x}]$, or (ii) $\underline{\mathsf{WR}}(x_0^{e-d} \cdot P_{1,2}^{[d]}) = O(e^5)$. We show that (i) is an impossibility while (ii) can happen only when $e \geq \exp(d)$.

Proof of Part (ii): Fix a random $x_0 = \alpha \in \mathbb{C}$. Note that, this implies that $P_{1,2}^{[d]} + \epsilon g = \sum_{i \in [k]} \ell_i^e$ for some affine forms $\widehat{\ell}_i \in \mathbb{C}[\epsilon^{\pm 1}][\boldsymbol{x}]$ and $g \in \mathbb{C}[\epsilon][\boldsymbol{x}]$ with $k \in O(e^5)$. Since $P_{1,2}^{[d]}$ is homogeneous, this also implies that $\underline{\mathsf{WR}}(P_{1,2}^{[d]}) \leq k$. But then Lemma 3.14 implies that $k \geq 2^{\Omega(d)}$, which in turn implies that $e \geq 2^{\Omega(d)}$.

Proof of Part (i): Let $x_0^{e-d} \cdot P_{1,2}^{[d]} = \prod_{i \in [e]} \ell_i + \ell_0^e$. The space of first order partials of the LHS has dimension at least $d + 2$, while the one of the RHS has dimension at most $e + 1$; since trivially $\prod_{i \in T} \ell_i$, for $T \subset [e]$, such that $|T| = e - 1$, and ℓ_0^{e-1} certainly span the space of single partial derivatives. Therefore, $e \geq d + 1$. This will be important since we will use the fact that $e - d \geq 1$, in the below.

Further, we can assume that $x_0 \nmid \ell_0$. Otherwise, say $\ell_0 = c \cdot x_0$, for some $c \in \mathbb{C}$, which implies that $x_0^{e-d} \mid \prod_{i \in [e]} \ell_i$. Hence, without loss of generality, we can assume that $\ell_i = x_0$, for $i \in [e - d]$ (we are assuming constants to be 1, because we can always rescale and push the constants to the other linear forms). Therefore, RHS is divisible by x_0^{e-d}. By dividing it out and renaming the linear forms appropriately, we get

$$P_{1,2}^{[d]} = \prod_{i \in [d]} \widehat{\ell}_i + c x_0^d \,,$$

where $\widehat{\ell}_i \in \mathbb{C}[\boldsymbol{x}]$. Further, we can put $x_0 = 0$. Note that, $x_0 \nmid \widehat{\ell}_i$, for any i, since otherwise x_0 divides RHS, but it doesn't divide the LHS. After substituting $x_0 = 0$, we get that

$$P_{1,2}^{[d]} = \prod_{i \in [d]} \tilde{\ell}_i \,,$$

where $\mathbb{C}[x_1, \ldots, x_{d+2}] \ni \tilde{\ell}_i = \widehat{\ell}_i|_{x_0=0} \neq 0$. From Lemma 3.12, it follows that this is not possible. A similar argument shows that $x_0 \nmid \ell_i$, for any $i \in [d]$; because otherwise that implies $x_0 \mid \ell_0$, and hence the above argument shows a contradiction.

Therefore, we assume that $x_0 \nmid \ell_i$, for $i \in [0, d]$. Now, there are two cases – (i) x_0 appears in ℓ_0, (ii) x_0 does not appear in ℓ_0.

If x_0 appears in ℓ_0, then say $\ell_0 = c_0 x_0 + \widehat{\ell}_0$, for some $c_0 \neq 0$. Note that $\widehat{\ell}_0 \in \mathbb{C}[x_1, \ldots, x_{d+2}]_1$ is non-zero, since we assume that $x_0 \nmid \ell_0$. Substitute $x_0 = -\widehat{\ell}_0 / c_0$ (so that ℓ_0 vanishes). This implies:

$$(-\widehat{\ell}_0 / c_0)^{e-d} \cdot P_{1,2}^{[d]} = \prod_{i \in [e]} \widehat{\ell}_i \,,$$

where $\widehat{\ell}_i = \ell_i|_{x_0 = -\widehat{\ell}_0/c_0}$. Since LHS is non-zero, so is each $\widehat{\ell}_i$. Since, everything is homogeneous, and we have unique factorization, the above implies that up to renaming, $P_{1,2}^{[d]} = c \cdot \prod_{i \in [d]} \widehat{\ell}_i$, which is a contradiction by Lemma 3.12.

If x_0 does not appear in ℓ_0, then there must exist an $i \in [e]$ such that x_0 appears in ℓ_i, otherwise RHS is x_0-free which is trivially a contradiction. We also know that x_0 cannot divide ℓ_i, by our assumption. So, say $\ell_i = c_i x_0 + \widehat{\ell}_i$, where $\widehat{\ell}_i$ is x_0-free, and $c_i \in \mathbb{C}$ is a nonzero element. Substitute $x_0 = -\widehat{\ell}_i / c_i$, so that ℓ_i vanishes. Since ℓ_0 is x_0-free, we immediately get that

$$(-\widehat{\ell}_0 / c_0)^{e-d} \cdot P_{1,2}^{[d]} = \ell_0^e \,.$$

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Again, by unique factorization, we get that $P_{1,2}^{[d]} = c \cdot \ell_0^d$, for some $c \in \mathbb{C}$, which is a contradiction by Lemma 3.12. This finishes the proof. $\qquad\square$

3.16 Theorem (Second exp. gap theorem). *If $P_{2,0}^{[d]} \trianglelefteq_{\mathrm{aff}} P_{1,2}^{[e]}$, then $e \geq \exp(d)$.*

Proof. Let $P_{2,0}^{[d]} \trianglelefteq_{\mathrm{aff}} P_{1,2}^{[e]}$. A similar formulation as above (in the previous theorem) gives us that $x_0^{e-d} \cdot P_{2,0}^{[d]} \trianglelefteq P_{1,2}^{[e]}$. By Theorem 3.11, we know that $x_0^{e-d} \cdot P_{2,0}^{[d]} \trianglelefteq P_{1,2}^{[e]}$ implies – either (i) $x_0^{e-d} \cdot P_{2,0}^{[d]} = g+h$, where $g = \prod_{i \in [e]} \ell_i$, for some linear forms $\ell_i \in \mathbb{C}[\boldsymbol{x}]_1$, and $\underline{\mathsf{WR}}(h) \leq 2$, or (ii) $\underline{\mathsf{WR}}(x_0^{e-d} \cdot P_{2,0}^{[d]}) = O(e^8)$. Similarly, as before, we show that (i) is an impossibility while (ii) can happen only when $e \geq \exp(d)$. Part (ii) proof is exactly to the argument in the proof of Theorem 3.15.

To prove the Part (i), there are two cases – (a) $h = \ell_0^e + \ell_{e+1}^e$, for $\ell_i \in \mathbb{C}[\boldsymbol{x}]_1$, or, (b) $h = \ell_0^{e-1} \cdot \ell_{e+1}$.

Case (a): Let $x_0^{e-d} \cdot P_{2,0}^{[d]} = \prod_{i \in [e]} \ell_i + \ell_0^e + \ell_{e+1}^e$. We assume that x_0 does not divide ℓ_i, for some $i \in \{0, e+1\}$, and each ℓ_i, for $i \in [e]$, otherwise, we can divide by the maximum power of x_0 on both the sides.

The space of first order partial derivatives of the LHS has dimension at least $2d$ whereas the one of the RHS has dimension at most $e+2$: since trivially $\prod_{i \in T} \ell_i$, for $T \subset [e]$, such that $|T| = e-1$, and $\ell_0^{e-1}, \ell_{e-1}^e$ certainly span the space of single partial derivatives. Therefore, $e \geq 2d - 2$.

Now, we divide this into subcases:

(a1) x_0 does not appear in ℓ_i, for any $i \in [e]$,
(a2) x_0 appears in ℓ_i, for some $i \in [e]$.

Case (a1): x_0 does not appear in ℓ_i, for $i \in [e]$. In that case, say $\ell_0 = c_0 x_0 + \widehat{\ell}_0$, and $\ell_{e+1} = c_{e+1} x_0 + \widehat{\ell}_{e+1}$, where $\widehat{\ell}_0$ and $\widehat{\ell}_{e+1}$ are x_0-free, and c_0, c_{e+1} are constants (might be 0 as well, but both cannot be 0 since then RHS becomes x_0-free). Therefore, the coefficient of x_0^{e-d} (as a polynomial) in RHS is $\gamma_0 \widehat{\ell}_0^d + \gamma_{e+1} \widehat{\ell}_{e+1}^d$, where $\gamma_0 = \binom{e}{d} c_0^{e-d}$, and similarly $\gamma_{e+1} = \binom{e}{d} c_{e+1}^{e-d}$. Comparing with LHS, we get that $P_{2,0}^{[d]} = \gamma_0 \widehat{\ell}_0^d + \gamma_{e+1} \widehat{\ell}_{e+1}^d$. Trivially, over $\mathbb{C}$, $\gamma_0 \widehat{\ell}_0^d + \gamma_{e+1} \widehat{\ell}_{e+1}^d$ is a product of linear forms, which is a contradiction, using Lemma 3.13.

Case (a2): If x_0 appears in one of the ℓ_i for $i \in [e]$, it can appear in two ways, either ℓ_i is a constant multiple of x_0, or $\ell_i = c_i x_0 + \widehat{\ell}_i$, where $\widehat{\ell}_i$ is a nonzero linear form which is x_0-free. Let $S_1 \subseteq [e]$ be such that $\ell_i = c_i \cdot x_0$, for $i \in S_1$, for some nonzero constant $c_i \in \mathbb{C}$, and $S_2 \subseteq [e]$ be such that $\ell_i = c_i x_0 + \widehat{\ell}_i$, where $\widehat{\ell}_i$ is nonzero.

Note that if $|S_1| + |S_2| < e - d$, then x_0^{e-d} cannot be contributed from the product and hence it only gets produced from $\ell_0^e + \ell_{e+1}^e$, and we get a contradiction in the same way as above. Hence, without loss of generality, assume that $|S_1| + |S_2| \geq e - d$.

If S_2 is non-empty, say $j \in S_2$, then substitute $x_0 = -\widehat{\ell}_j/c_j$, so that ℓ_j becomes 0. This substitution gives us the following:

$$(-\widehat{\ell}_j/c_j)^{e-d} \cdot P_{2,0}^{[d]} \;=\; \tilde{\ell}_0^e + \tilde{\ell}_{e+1}^e \;.$$

Since, $\tilde{\ell}_0^e + \tilde{\ell}_{e+1}^e$ can be written as a product of linear forms, from the unique factorization, it follows that f must be a product of linear forms, which is a contradiction from Lemma 3.13. Hence, we are done when $|S_2|$ is non-empty.

If S_2 is empty, since $|S_1| + |S_2| \geq e - d$ by assumption, we have $|S_1| \geq e - d$. In particular, $x_0^{e-d} \mid$ LHS $- \prod \ell_i \implies x_0^{e-d} \mid \ell_0^e + \ell_{e+1}^e = \prod_i (\ell_0 - \zeta^{2i+1} \ell_{e+1})$, where ζ is $2e$-th root of unity. Since, $e - d \geq 2$ for $d \geq 4$, this simply implies that there are two indices i_1 and i_2 such that $\ell_0 - \zeta^{i_1} \ell_{e+1} = c_{i_1} x_0$, and $\ell_0 - \zeta^{i_2} \ell_{e+1} = c_{i_2} x_0$. Together, this implies that both ℓ_0 and ℓ_{e+1} are multiples of x_0, which is a contradiction, since we assumed that x_0 cannot divide each ℓ_i, for $i \in [0, e+1]$. Hence, we are done with case (a).

Case (b): Let $x_0^{e-d} \cdot P_{2,0}^{[d]} = \prod_{i \in [e]} \ell_i + \ell_0^{e-1} \cdot \ell_{e+1}$. We assume that x_0 does not divide both ℓ_i, for some $i \in [e]$, and one of the ℓ_0 or ℓ_{e+1}, otherwise, we can divide by the maximum power x_0 both side. Again, a similar argument shows that $e \geq 2d - 2$.

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Similarly, as before, we divide into subcases: (b1) x_0 does not appear in ℓ_i, for any $i \in [e]$, (b2) x_0 appears in ℓ_i, for some $i \in [e]$.

Case (b1): If x_0 does not appear in the first product, i.e,. any of ℓ_i, for $i \in [e]$, then it must appear in ℓ_0 (because if it only appears in ℓ_{e+1}, the degree of x_0 is 1 in RHS, a contradiction). Note that, $x_0 \nmid \ell_0$ (and similarly ℓ_{e+1}), because otherwise, substituting $x_0 = 0$ makes LHS 0, while RHS remains $\prod_{i \in [e]} \ell_i$. Hence, let $\ell_0 := c_0 x_0 + \widehat{\ell}_0$, where $\widehat{\ell}_0$ is x_0-free. Substitute $x_0 = -\widehat{\ell}_0/c_0$, so that

$$(-\widehat{\ell}_0/c_0)^{e-d} \cdot P_{2,0}^{[d]} \; = \; \prod_{i \in [e]} \ell_i \ .$$

This in particular implies that $P_{2,0}^{[d]}$ is a product of linear forms, which is a contradiction by Lemma 3.13.

Case (b2): In this case, without loss of generality, x_0 appears in ℓ_1. Note that, x_0 cannot divide ℓ_1, because otherwise, it must divide LHS-$\prod_{i \in [e]} = \ell_0^{e-1}\ell_{e+1}$, which implies that x_0 must divide one of the ℓ_0 or ℓ_{e+1}, contradicting the minimality of x_0-division. Therefore, $\ell_1 = c_1 x_0 + \widehat{\ell}_1$, where c_1 is a nonzero constant, and $\widehat{\ell}_1$ is a nonzero linear form which is x_0-free. Substitute $x_0 = -\widehat{\ell}_1/c_1$, both side to get that

$$(-\widehat{\ell}_1/c_1)^{e-d} \cdot P_{2,0}^{[d]} \; = \; \widehat{\ell}_0^{e-1}\widehat{\ell}_{e+1} \ .$$

Therefore, again by unique factorization, we get that f must a product of linear forms, which is a contradiction by Lemma 3.13.

$\square$

4 Geometric complexity theory for product-plus-power

In this section, we study computational and invariant theoretic properties of the polynomial

$$P_{r,s}^{[d]} = \sum_{k=1}^{r}\prod_{i=1}^{d} x_{ki} + \sum_{j=1}^{s} y_j^d,$$

defined in Section 1.3; this is a polynomial of degree d in $rd + s$ variables. The cases $(r,s) = (1,1),(1,2),(2,0),(0,r)$ correspond, respectively, to the product-plus-power, the product-plus-two-powers, the binomial, and the power sum polynomial mentioned in the previous sections.

Theorem 4.2 determines the stabilizer of $P_{r,s}^{[d]}$ under the action of the group GL_{rd+s} acting on the variables. The knowledge of the stabilizer $H = \mathrm{Stab}_{\mathrm{GL}_{d+1}}(P_{1,1}^{[d]})$ allows us to determine the representation theoretic structure of the coordinate ring of the orbit of $P_{1,1}^{[d]}$, which is achieved in Proposition 4.4. Recall that the irreducible representations of GL_{d+1} are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \ldots$, with $\ell(\lambda) \leq d+1$, see Section 4.2. Denote by $S_\lambda(\mathbb{C}^{d+1})$ the irreducible representation of type λ. In this section we write $S^d V = S_{(d)} V$ for the space of homogeneous polynomials of degree d in $\dim V$ variables. For every integer D and every partition λ of dD, we obtain the following identity:

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} P_{1,1}^{[d]}]) \; = \; \dim(S_\lambda \mathbb{C}^{d+1})^H \; = \; \sum_{\delta=0}^{D} \sum_{\mu \vdash \delta d, \mu \preceq \lambda, \ell(\mu) \leq d} a_\mu(d,\delta)\,,$$

where $a_\mu(d,\delta)$ is the *plethysm coefficients*, that is the multiplicity of $S_\mu(\mathbb{C}^{d+1})$ in $S^d(S^\delta(V))$, see Proposition 4.4. We use this formula, and apply the [IK20] approach to lower bounds on $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} P_{0,d}^{[d]}}])$, to find a sequence of partitions where $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} P_{1,1}^{[d]}}]) <$ $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} P_{0,d}^{[d]}}])$, see Theorem 4.10. This implies $\overline{\mathrm{GL}_{d+1} P_{0,d}^{[d]}} \not\subseteq \overline{\mathrm{GL}_{d+1} P_{1,1}^{[d]}}$ if $d \geq 3$.

We implement this approach explicitly and we determine via a computer calculation an abundance of multiplicity obstructions against generic polynomials, see Appendix A.

In Proposition 4.5, we prove that $P_{r,s}^{[d]}$ is polystable, in the sense of invariant theory. This guarantees the existence of a *fundamental invariant*, in the sense of [BI17]: in Proposition 4.7, we show a connection between the degree of this fundamental invariant and the Alon-Tarsi conjecture on Latin squares in combinatorics.

4.1 Stabilizer

Consider the action of the general linear group GL_n on the homogeneous components of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, by linear change of variables, as described in Section 1.3. For a homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]_d$, write $\mathrm{Stab}_{GL_n}(f)$ for its stabilizer under this action. It is an immediate fact that $\mathrm{Stab}_{GL_n}(f)$ is a closed algebraic subgroup of GL_n. It may consists of several connected (irreducible) components: the *identity component*, denoted $\mathrm{Stab}_{GL_n}^0(f)$ is the connected component containing the identity; $\mathrm{Stab}_{GL_n}^0(f)$ is a closed, normal subgroup of $\mathrm{Stab}_{GL_n}(f)$ [Ges16, Lemma 2.1]; the quotient $\mathrm{Stab}_{GL_n}(f)/\mathrm{Stab}_{GL_n}^0(f)$ is a finite group.

The Lie algebra $\mathfrak{g}$ of an algebraic group G can be geometrically identified with the tangent space to G at the identity element. Moreover, if G is a subgroup of GL_n, then $\mathfrak{g}$ is naturally a subalgebra of $\mathfrak{gl}_n = \mathrm{End}(\mathbb{C}^n)$; moreover $\mathfrak{g}$ uniquely determined the identity component of G.

It is a classical fact that the Lie algebra of $\mathrm{Stab}_{GL_n}(f)$ is the annihilator of f under the Lie algebra action of $\mathfrak{gl}_n$ on $\mathbb{C}[\mathbf{x}]_d$; denote this annihilator by $\mathfrak{ann}_{\mathfrak{gl}_n}(f)$. Typically, in order to determine $\mathrm{Stab}_{GL_n}(f)$, one first computes $\mathfrak{ann}_{\mathfrak{gl}_n}(f)$, which uniquely determines $\mathrm{Stab}_{GL_n}^0(f)$. Then, one determines $\mathrm{Stab}_{GL_n}(f)$ as a subgroup of the normalizer $N_{GL_n}\mathrm{Stab}_{GL_n}^0(f)$. The last step is often challenging; some methods to do this systematically in simplified settings are presented in [GG15, Ges16, GIP17].

First, we record a general result regarding the stabilizer of sums of polynomials in disjoint sets of variables. This is the symmetric version of [CGL+21, Thm. 4.1(i)]. We say that a polynomial $f \in S^d \mathbb{C}^n$ is *concise* (in $S^d \mathbb{C}^n$) if the first order partials of f are linearly independent.

4.1 Lemma. *Let $V = V_1 \oplus V_2$ and let $f \in \mathbb{C}[V^*]_d = S^d V$ be a homogeneous polynomial with $f = f_1 + f_2$, where $f_i \in S^d V_i$ are both concise, with $d \geq 3$. Then*

(i) $\mathfrak{ann}_{\mathfrak{gl}(V)}(f_1) = \mathfrak{ann}_{\mathfrak{gl}(V_1)}(f_1) \oplus \mathrm{Hom}(V_2, V)$;

(ii) $\mathfrak{ann}_{\mathfrak{gl}(V)}(f_1 + f_2) = \mathfrak{ann}_{\mathfrak{gl}(V_1)}(f_1) \oplus \mathfrak{ann}_{\mathfrak{gl}(V_2)}(f_2)$.

Proof. For both statements, the inclusion of the right-hand term into the left-hand term is clear. We prove the reverse inclusion.

For $X \in \mathfrak{gl}(V)$, write $X = \sum_{i,j=1}^2 X_{ij}$, with $X_{ij} \in \mathrm{Hom}(V_i, V_j)$.

The proof of (i) amounts to showing that if $X \in \mathfrak{ann}_{\mathfrak{gl}(V)}(f_1)$, then $X_{12} = 0$ and $X_{11} \in \mathfrak{ann}_{\mathfrak{gl}(V_1)}(f_1)$. Suppose $X.f_1 = 0$. Notice $X.f_1 = X_{11}.f_1 + X_{12}.f_1$; here $X_{11}.f_1 \in S^d V_1$ and $X_{12}.f_1 \in V_2 \otimes S^{d-1} V_1$. In particular, both terms must vanish. The term $X_{12}.f_1$ is a sum of at most $\dim V_2$ linearly independent elements, each of which is a linear combination of first order partials of f_1. Therefore, all such linear combinations must be 0, and since f_1 is concise in $S^d V_1$, we obtain $X_{12} = 0$. The condition $X_{11}.f_1 = 0$ is, by definition, equivalent to $X_{11} \in \mathfrak{ann}_{\mathfrak{gl}(V_1)}(f_1)$. This conclude the proof of (i).

To prove (ii), we show that if $X \in \mathfrak{ann}_{\mathfrak{gl}(V)}(f)$, then $X_{12} = 0$, $X_{21} = 0$ and $X_{ii} \in \mathfrak{ann}_{\mathfrak{gl}(V_i)}(f_i)$. Suppose $X.f = 0$. We have $X.f = (X_{11} + X_{12}).f_1 + (X_{21} + X_{22}).f_2$. Notice

$$\begin{array}{ll} X_{11}f_1 \in S^d V_1, & X_{12}f_1 \in V_2 \otimes S^{d-1} V_1, \\ X_{21}f_2 \in V_1 \otimes S^{d-1} V_2, & X_{22}f_2 \in S^d V_2. \end{array}$$

Since $d \geq 3$, the four terms are linearly independent, hence they all must vanish individually. This concludes the proof. $\square$

We can now determine the stabilizer of $P_{r,s}^{[d]}$. Let $\mathbb{T}^{\mathrm{SL}_n}$ denote the subgroup of diagonal elements in SL_n. We use the *wreath product* notation: given a group G, the wreath product $G \wr \mathfrak{S}_k$ for the semidirect product $G^{\times k} \rtimes \mathfrak{S}_k$ where $\mathfrak{S}_k$ acts on the direct product $G^{\times k}$ by permuting the direct factors; we refer to [Rob96, Sec. 1.6] for details on this construction.

4.2 Theorem. *For $d \geq 3$ and for every r, s, we have*

$$\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]}) = ([\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d] \wr \mathfrak{S}_r) \times (\mathbb{Z}_d \wr \mathfrak{S}_s);$$

each copy of $\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d$ acts by rescaling and permuting the variables in one of the r sets $\{x_{ji} : i = 1, \ldots, d\}$ for $j = 1, \ldots, r$; the group $\mathfrak{S}_r$ permutes (set-wise) these sets; the group $\mathbb{Z}_d \wr \mathfrak{S}_s$ acts by rescaling (by a d-th root of 1) and permuting the variables in the set $\{y_i : i = 1, \ldots, s\}$.

Proof. It is clear that the group on the right-hand side is contained in the stabilizer $\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]})$. We show the reverse inclusion.

First, we determine the identity component of $\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]})$. By Lemma 4.1, the annihilator of $P_{r,s}^{[d]}$ in $\mathfrak{gl}(V)$ is the direct sum of the annihilators of its summands. This guarantees that the identity component of the stabilizer is the product of the identity components for the summands of $P_{r,s}^{[d]}$. The identity component for each square-free monomial is a copy of $\mathbb{T}^{\mathrm{SL}_d}$, see, e.g., [Lan17, Sec. 7.1.2]. The identity component of each power is trivial. Therefore, we deduce $\mathrm{Stab}_{\mathrm{GL}(V)}^0(P_{r,s}^{[d]}) = (\mathbb{T}^{\mathrm{SL}_d})^{\times r}$.

Since $\mathrm{Stab}_{\mathrm{GL}(V)}^0(P_{r,s}^{[d]})$ is a normal subgroup of $\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]})$, we have

$$\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]}) \subseteq N_{\mathrm{GL}_{rd+s}}(\mathbb{T}^{\mathrm{SL}_d \times r}) = ([\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d] \wr \mathfrak{S}_r) \rtimes Q$$

where Q is the parabolic subgroup stabilizing the subspace spanned by the x_{ij} variables; here $N_{\mathrm{GL}_{rd+s}}(=)$ denotes the normalizer subgroup.

In order to determine the discrete component, we follow the same argument as the one used for the power sum polynomial $P_{0,s}^{[d]}$ in [Lan17, Section 8.12.1]. In particular, $\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]})$ stabilizes the Hessian determinant of $P_{r,s}^{[d]}$, up to scaling. A direct calculation shows that this Hessian determinant, up to scaling, is

$$H = (\prod_{i,j} x_{ij} \prod_k y_k)^{d-2}.$$

Unique factorization implies that $\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]}) \cap Q \subseteq \mathbb{T} \rtimes \mathfrak{S}_s$, where $\mathbb{T}$ is the torus of diagonal matrices acting on the y_j variables. Hence this subgroup commutes with $[\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d] \wr \mathfrak{S}_r$ and we deduce

$$\mathrm{Stab}_{\mathrm{GL}(V)}(P_{r,s}^{[d]}) \cap Q = \mathrm{Stab}_{\mathrm{GL}_s}(y_1^d + \cdots + y_s^d) = \mathbb{Z}_d \wr \mathfrak{S}_s.$$

This concludes the proof. $\qquad\square$

In the context of geometric complexity theory it is important to know if the polynomial is characterized by its stabilizer [MS08]. While this property fails for the polynomials $P_{r,s}^{[d]}$, a slightly weaker statement is true — every polynomial stabilized by $\mathrm{Stab}(P_{r,s}^{[d]})$ is in the orbit of $P_{r,s}^{[d]}$ or a very special restriction of $P_{r,s}^{[d]}$. This is similar to the properties of minrank tensors and slice rank tensors considered in [BIL$^+$19].

4.3 Theorem. *If a polynomial $f \in \mathbb{C}[x_{11}, \ldots, x_{dr}, y_1, \ldots, y_s]_d$ is stabilized by $\mathrm{Stab}(P_{r,s}^{[d]})$, then*

$$f = \alpha \sum_{i=1}^{r} \prod_{j=1}^{d} x_{ji} + \beta \sum_{i=1}^{s} y_i^d,$$

for some $\alpha, \beta \in \mathbb{C}$.

Proof. Partition the set of variables into the subsets $X_i = \{x_{1i}, \ldots, x_{di}\}$ and $Y_i = \{y_i\}$. Note that $\mathrm{Stab}(P_{r,s}^{[d]})$ contains the transformation which scales all variables in one of the subsets by a d-th root of unity, acting as identity on all other variables. It follows that each monomial of f contains variables from only one of the subsets, because if this was not the case then the transformation described above multiplies the monomial by a coefficient different from 1. Thus we have
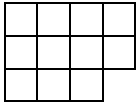
$$f = \sum_{i=1}^{r} f_i(x_{1i}, \ldots, x_{di}) + \sum_{i=1}^{s} \beta_i y_i^d.$$

Since f is fixed under the symmetric group $\mathfrak{S}_s$ permuting $y_1, \ldots, y_s$, the coefficients β_i are all equal. Since f is fixed under the symmetric group $\mathfrak{S}_r$ permuting the subsets $X_1, \ldots, X_r$, all the polynomials f_i also coincide.

Finally, the stabilizer group contains the transformations scaling x_{j1} by λ and x_{k1} by λ^{-1}. This transformation scales a monomial $x_{j1}^{p_j} x_{k1}^{p_k} \ldots$ by $\lambda^{p_j - p_k}$. It follows that each monomial of f_1 must have the same degree with respect to each variable, that is, $f_1 = \alpha \prod_{j=1}^{d} x_{j1}$. $\qquad\square$

4.2 Multiplicities in the coordinate ring of the orbit

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a finite non-increasing sequence of nonnegative integers. We write $\ell(\lambda) := \max\{i \mid \lambda_i \neq 0\}$, and $\lambda \vdash D$ means $\sum_i \lambda_i = D$. To each partition λ we associate its Young diagram, which is a top-left justified array of boxes with λ_i boxes in row i. For example, the Young diagram of $\lambda = (4, 4, 3)$ is [diagram]. The transpose of the Young diagram is obtained by switching rows and columns. Denote the partition corresponding to this Young diagram by λ^t, for example $(4, 4, 3)^t = (3, 3, 3, 2)$. A group homomorphism $\varrho : \mathrm{GL}_D \to \mathrm{GL}(V)$, where V is a finite dimensional complex vector space, is called a *representation* of GL_D. A representation is *polynomial* if each entry of the matrix corresponding to the linear map $\varrho(g)$ is given by a polynomial in the entries of the elements of GL_D. A linear subspace that is closed under the group operation is called a *subrepresentation*. A representation with only the two trivial subrepresentations is called *irreducible*. The irreducible polynomial representations of GL_{d+1} are indexed by partitions λ with $\ell(\lambda) \leq d + 1$, see for example [Ful97, Ch. 8]. Denote by $S_\lambda(\mathbb{C}^{d+1})$ the irreducible representation of type λ. For a GL_{d+1}-representation V, write $\mathrm{mult}_\lambda(V)$ to denote the multiplicity of λ in V, i.e., the dimension of the space of equivariant maps from $S_\lambda(\mathbb{C}^{d+1})$ to V, or equivalently, the number of summands of isomorphism type λ in any decomposition of V into a direct sum of irreducible representations.

In this section, we consider the representations given by the homogeneous components of the coordinate ring of $\mathrm{GL}_{d+1} \cdot P_{1,1}^{[d]}$. By Theorem 4.2, the stabilizer of $P_{1,1}^{[d]}$ under the action of GL_{d+1} is $H := \mathrm{Stab}_{\mathrm{GL}_{d+1}}(P_{1,1}^{[d]}) \simeq \mathbb{Z}_d \times (\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d)$.

The stabilizer is used to determine the multiplicities in the coordinate ring of the group orbit $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} P_{1,1}^{[d]}])$. It is a general fact that $\mathbb{C}[\mathrm{GL}_{d+1} P_{1,1}^{[d]}] \simeq \mathbb{C}[\mathrm{GL}_{d+1}]^H$: in other words the coordinate ring of the orbit coincides with the subring of H-invariant elements in the coordinate ring of the group GL_{d+1}. This ring of H-invariants can be determined using the Algebraic Peter-Weyl Theorem [GW09, Thm. 4.2.7], a powerful tool that allows us to compute the relevant multiplicities. In turn, we have

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} P_{1,1}^{[d]}]) = \dim(S_\lambda \mathbb{C}^{d+1})^H.$$

We determine the dimension of these invariant spaces by classical representation branching rules, see Proposition 4.4.

For partitions μ and λ, define $\mu \preceq \lambda$ if and only if the Young diagram of μ is contained in the one of λ, (that is for every i, one has $\mu_i \leq \lambda_i$) and the skew diagram λ/μ given by the difference has at most 1 box in each column (that is $\lambda_i^t - \mu_i^t \leq 1$). In other words, λ can be obtained from μ by adding a suitable number of boxes, with no two of them in the same column. The plethysm

coefficient is defined to be the multiplicity $a_\mu(d, D) := \mathrm{mult}_\mu(S^d(S^D(\mathbb{C}^N)))$, and it does not depend on N as long as $N \geq d$.

4.4 Proposition. *For $\lambda \vdash dD$ we have*

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1}\, P_{1,1}^{[d]}]) = \dim(S_\lambda \mathbb{C}^{d+1})^H = \sum_{\delta=0}^{D} \sum_{\substack{\mu \vdash \delta d \\ \mu \preceq \lambda \\ \ell(\mu) \leq d}} a_\mu(d, \delta).$$

Proof.

$$(S_\lambda \mathbb{C}^{d+1})^H = (S_\lambda(\mathbb{C} \oplus \mathbb{C}^d) \downarrow_{\mathrm{GL}_1 \times \mathrm{GL}_d}^{\mathrm{GL}_{d+1}})^{\mathbb{Z}_d \times (\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d)} \overset{\text{Pieri's rule}}{=} \bigoplus_{\substack{\mu \preceq \lambda \\ \ell(\mu) \leq d}} (S^{|\lambda|-|\mu|}\mathbb{C}^1)^{\mathbb{Z}_d} \otimes (S_\mu \mathbb{C}^d)^{\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d},$$

where Pieri's rule is a well-known decomposition rule, see for example [FH91, p. 80, Exe. 6.12]. Now, $\dim((S^{|\lambda|-|\mu|}\mathbb{C}^1)^{\mathbb{Z}_d}) = 1$ if and only if $|\lambda| - |\mu|$ is a multiple of d if and only if $|\mu|$ is a multiple of d. Otherwise it is 0. Hence

$$\dim(S_\lambda V)^H = \sum_{\delta=0}^{d} \sum_{\substack{\mu \vdash \delta d \\ \mu \preceq \lambda \\ \ell(\mu) \leq d}} \underbrace{\dim(S_\mu \mathbb{C}^d)^{\mathbb{T}^{\mathrm{SL}_d} \rtimes \mathfrak{S}_d}}_{=a_\mu(d, \delta)}.$$

The last underbrace equality is Gay's theorem [Gay76]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The condition that $\ell(\mu) \leq d$ is not necessary, because if $\ell(\mu) > d$, then $a_\mu(d, \delta) = 0$.

A computer calculation shows that this indeed gives multiplicity obstructions in the sense of Section 1.3. We record the result of the this calculation in Appendix A. We used the HWV software [BHIM22] to directly calculate that $(10, 6, 4, 4)$ and $(8, 8, 4, 4)$ are the only types in the vanishing ideal for $D = 8$, $d = 3$. For $d = 3$ there are no equations in degree $1, \ldots, 7$. In particular, none of Brill's equations from [Gor94], which are of degree $d + 1$, vanishes on $\mathrm{GL}_{d+1}\, P_{1,1}^{[d]} \cap S^d \mathbb{C}^d$.

4.3 Polystability

A polynomial $f \in S^d V$ is called *polystable* if its $\mathrm{SL}(V)$-orbit is closed. Polystability is an important property in GCT, as it implies the existence of a *fundamental invariant* that connects the GL-orbit with the GL-orbit closure, see [BI17, Def. 3.9 and Prop. 3.10]. This connection can be used to exhibit multiplicity obstructions, as was done in [IK20].

4.5 Proposition. *Let $d \geq 2$. The polynomial $P_{r,s}^{[d]}$ is polystable, i.e., the orbit $\mathrm{SL}_{rd+s}\, P_{r,s}^{[d]}$ is closed.*

Proof. If $d = 2$, then $P_{r,s}^{[2]}$ is a polynomial of degree 2 defining a quadratic form of maximal rank. This is polystable.

Suppose $d \geq 3$. A criterion for polystability is given in [BI17, Prop. 2.8], based on works of Hilbert, Mumford, Luna, and Kempf.

In order to apply this criterion, consider the group $R = \mathrm{Stab}(P_{r,s}^{[d]}) \cap \mathbb{T}$, where $\mathbb{T}$ denotes the torus of diagonal matrices in GL_{rd+s}, in the basis defined by the variables. By Theorem 4.2, we deduce $R = (\mathbb{T}^{\mathrm{SL}_d})^{\times r} \times \mathbb{Z}_d^{\times s}$. This is a group consisting entirely of diagonal matrices and it is easy to verify that its centralizer in SL_{d+1} coincides with $\mathbb{T}^{\mathrm{SL}_{d+1}}$. This proves the first property of the criterion.

For the second property, consider the exponent vectors of the monomials appearing in $P_{r,s}^{[d]}$. For a monomial m, write $\mathrm{wt}(m)$ for its exponent vector. It is immediate to verify that

$$\sum_{i=1}^{r} \mathrm{wt}(x_{i1} \cdots x_{id}) + \frac{1}{d} \sum_{j=1}^{s} \mathrm{wt}(y_j^d) = (1, \ldots, 1);$$

this shows that the vector $(1, \ldots, 1)$ lies in the convex cone generated by the exponent vectors of the monomials of $P_{r,s}^{[d]}$. This proves the second part of the criterion and concludes the proof. $\square$

Proposition 4.5 reduces to the following in the special case $r = s = 1$:

4.6 Corollary. *Let $d \geq 2$. The product-plus-power polynomial $P_{1,1}^{[d]}$ is polystable, i.e., the orbit $\mathrm{SL}_{d+1} \, P_{1,1}^{[d]}$ is closed.*

4.4 Fundamental invariants and the Alon-Tarsi conjecture

The *fundamental invariant* Φ of a polystable polynomial $f \in S^D V$ is the unique (up to scaling) smallest degree $\mathrm{SL}(V)$-invariant function in $\mathbb{C}[\overline{\mathrm{GL}(V)f}]$, see Def. 3.8 in [BI17]. It describes the connection between the orbit and the orbit-closure of f: more formally, the coordinate ring of the orbit $\mathbb{C}[\mathrm{GL}(V)f]$ is canonically isomorphic to the localization at Φ of the coordinate ring of the orbit-closure, that is $\mathbb{C}[\overline{\mathrm{GL}(V)f}]_\Phi$; see [BI17, Pro. 3.9]. This connection can be used to exhibit multiplicity obstructions, as was done in [IK20].

It is known that for even d the orbit closure $\overline{\mathrm{GL}_d(x_1 \cdots x_d)}$ of a squarefree monomial has fundamental invariant of degree d if and only if the Alon-Tarsi conjecture on Latin squares holds for d; see [KL15] and [BI17, Prop. 3.26]; otherwise the fundamental invariant has higher degree. In this section we show an analogous result for the orbit closure $\overline{\mathrm{GL}_{d+1}(x_1 \cdots x_d + x_{d+1}^d)}$: if d is even this orbit closure has fundamental invariant of degree $d + 1$ if and only if the Alon-Tarsi conjecture on Latin squares holds for d; otherwise the fundamental invariant has higher degree.

4.7 Proposition. *Let d be even. The degree of the fundamental invariant of $P_{1,1}^{[d]}$ is $d + 1$ if and only if the Alon-Tarsi conjecture for d is true, otherwise it is of higher degree.*

Proof. We follow the presentation in [CIM17, BI17, BDI21]. For a partition λ we place positive integers into the boxes of the Young diagram and call it a *tableau* T of shape λ. The vector of numbers of occurrences of 1s, 2s, etc, is called the *content* of T. The content is $n \times d$ if T has exactly d many 1s, d many 2s, ..., d many ns. The set of boxes of the Young diagram of λ is denoted by $\mathrm{boxes}(\lambda)$. The boxes that have the same number are said to form a *block*.

Let $m = n + 1$. Fix a tableau T of shape λ with content $n \times d$ and fix a tensor $p = \sum_{i=1}^{r} \ell_{i,1} \otimes \cdots \otimes \ell_{i,d} \in \otimes^d \mathbb{C}^m$. A placement
$$\vartheta : \mathrm{boxes}(\lambda) \to [r] \times [d]$$
is called *proper* if the first coordinate of ϑ is constant in each block and the second coordinate of ϑ in each block is a permutation. We define the determinant of a matrix that has more rows than columns as the determinant of its largest top square submatrix.

For a tableau T with content $\Delta \times d$ we define the polynomial f_T via its evaluation on p:

$$f_T(p) := \sum_{\text{proper } \vartheta} \prod_{c=1}^{\lambda_1} \det_{\vartheta,c} \quad \text{with} \quad \det_{\vartheta,c} := \det \left(\ell_{\vartheta(1,c)} \ldots \ell_{\vartheta(\mu_c,c)} \right) \tag{4.8}$$

The degree of f_T is Δ. The polynomial f_T is SL_m-invariant if and only if the shape of T is rectangular with exactly m many rows. It is easy to see that $f_T = 0$ if T has any column in which a number appears more than once. Moreover, it is easy to see that f_T is fixed (up to sign) when two entries in T are exchanged within a column. So, up to sign, there is only one T that could give an SL_m-invariant of degree $d + 1$: It is the tableau with $m = d + 1$ many rows and d columns that has only entries i in row i. For $n = 4$ it looks as follows.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 4 & 4 & 4 & 4 \\ \hline 5 & 5 & 5 & 5 \\ \hline \end{array}$$

For this T it remains to verify that f_T does not vanish identically on to orbit closure $\overline{\mathrm{GL}_{d+1}(x_1 \cdots x_d + x_{d+1}^d)}$. Since f_T is SL_{d+1}-invariant, this is equivalent to f_T not vanishing at the point $x_1 \cdots x_d + x_{d+1}^d$. So we now evaluate $f_T(x_1 \cdots x_d + x_{d+1}^d)$. The nonzero summands in Equation (4.8) must place $(d+1, *)$ into one of the blocks. We can partition the summands according to the row in which $(d+1, *)$ is placed. Since the number of columns is even, each part of the partition contributes the same number to the overall sum. That number is the column sign of the unique Latin square that is obtained when removing the row in which $(d+1, *)$ is placed. Hence the whole sum if $d+1$ times the difference of the column-even and column-odd Latin squares, so its nonvanishing is equivalent to the Alon-Tarsi conjecture for d. $\qquad\square$

4.9 Remark. Other fundamental invariants connected to the Alon-Tarsi conjecture have recently been studied in [LZX21, AY22].

4.5 New obstructions

For two partitions λ and μ, their sum is defined coordinatewise, i.e., $(\lambda + \mu)_i = \lambda_i + \mu_i$. We write $a \times b$ for the partition $(b, b, \ldots, b)$ of ab. For example, if $d = 3$, then the Young diagram to $\lambda := (5d - 1, 1) + ((d+1) \times (10d))$ is the following:
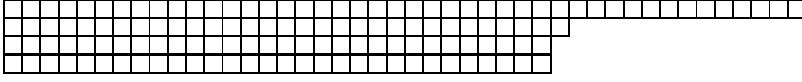
This section is devoted to proving the following result, which is a restatement of Theorem 1.4.

4.10 Theorem. Let $d \geq 3$ be even, and let $\lambda := (5d - 1, 1) + \big((d+1) \times (10d)\big)$. Then we have representation theoretic multiplicity obstructions:

$$\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}}]) \leq 4 < 5 = \mathrm{mult}_\lambda(\overline{\mathbb{C}[\mathrm{GL}_{d+1}(x_1^d + \cdots + x_{d+1}^d)]}),$$

and hence $\overline{\mathrm{GL}_{d+1}(x_1^d + \cdots + x_{d+1}^d)} \not\subseteq \overline{\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}}$.

We point out that these obstructions of Theorem 4.10 are only based on the symmetries of the two polynomials as in [IK20].

The upper and the lower bound are proved independently, see Proposition 4.11 and Proposition 4.12, which proves the theorem. Fix the following notation: $\kappa := (5d - 1, 1)$, $\blacksquare := (d+1) \times (10d)$, $\square := d \times (10d)$, $\Delta := |\square|/d = 10d$, and $\lambda := \kappa + \blacksquare$.

4.11 Proposition. $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}}]) \leq \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}]) = \mathrm{mult}_\kappa(\mathbb{C}[\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}]) = 4$.

Proof. The ring $\mathbb{C}[\overline{\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}}]$ is a localization of the ring $\mathbb{C}[\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}]$, see [BI17], which implies $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}}]) \leq \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}])$. We observe that $a_{\nu+\square}(d, i+\Delta) = a_\nu(d, i)$, because $\square$ has an even number of columns and exactly d rows. Then we calculate:

$$
\begin{aligned}
\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_{d+1} \, P_{1,1}^{[d]}]) &= \sum_{\delta=0}^{5} \sum_{\substack{\mu \vdash \delta d \\ \mu \preceq \lambda \\ \ell(\mu) \leq d}} a_\mu(d, \delta) \\
&= a_{(d)+\square}(d, 1+\Delta) + a_{(d-1,1)+\square}(d, 1+\Delta) + a_{(2d)+\square}(d, 2+\Delta) \\
&\quad + a_{(2d-1,1)+\square}(d, 2+\Delta) + a_{(3d)+\square}(d, 3+\Delta) + a_{(3d-1,1)+\square}(d, 3+\Delta) \\
&\quad + a_{(4d)+\square}(d, 4+\Delta) + a_{(4d-1,1)+\square}(d, 4+\Delta) + a_{(5d-1,1)+\square}(d, 5+\Delta) \\
&= a_{(d)}(d, 1) + a_{(d-1,1)}(d, 1) + a_{(2d)}(d, 2) + a_{(2d-1,1)}(d, 2) + a_{(3d)}(d, 3) \\
&\quad + a_{(3d-1,1)}(d, 3) + a_{(4d)}(d, 4) + a_{(4d-1,1)}(d, 4) + a_{(5d-1,1)}(d, 5) \\
&= 4,
\end{aligned}
$$

because $a_{(nm)}(n, m) = 1$, and $a_{(nm-1,1)}(n, m) = 0$, because $(nm - 1, 1)$ is of hook shape, see [BI18, Prop. 19.3.20]. Note that there is no summand $a_{(5d)}(d, 5)$ and no summand $a_{(0)}(d, 0)$, because $(5d) \not\preceq (5d - 1, 1)$, and $(0) \not\preceq (5d - 1, 1)$. $\qquad\square$

4.12 Proposition. $\text{mult}_\lambda(\mathbb{C}[\overline{\text{GL}_{d+1}(x_1^d + \cdots + x_{d+1}^d)}]) \geq 5$.

Proof. We use the Main Technical Theorem 4.2 from [IK20]. Consider all partitions ϱ of 5, and observe that $\sum_{i=1}^{d+1} 2\lceil \frac{\varrho_i}{2(d-2)} \rceil \leq 10$. In the notation of [IK20], we set $e_\Xi := 10$, which is exactly how many $(d + 1) \times d$ blocks form $\blacksquare$.

For a partition $\varrho \vdash_m D$ the *frequency notation* $\widehat{\varrho} \in \mathbb{N}^m$ is defined via $\widehat{\varrho}_i := |\{j \mid \varrho_j = i\}|$. For example, the frequency notation of $\varrho = (3, 3, 2, 0)$ is $\widehat{\varrho} = (0, 1, 2, 0)$. We observe that $|\varrho| = \sum_i i\widehat{\varrho}_i$. We first use Theorem 4.1 from [IK20] (with adjusted notation):

Let $m := d + 1$, $D := 5$, $\kappa = (5d - 1, 1) \vdash_m Dd$. Define

$$b(\kappa, \varrho, d, D) := \sum_{\substack{\mu^1, \mu^2, \ldots, \mu^D \\ \mu^i \vdash di\widehat{\varrho}_i}} c_{\mu^1, \mu^2, \ldots, \mu^D}^\kappa \prod_{i=1}^D a_{\mu^i}(\widehat{\varrho}_i, i \cdot d).$$

Then

$$\text{mult}_\kappa \mathbb{C}[\text{GL}_m(x_1^d + x_2^d + \cdots + x_m^d)] = \sum_{\varrho \vdash_m D} b(\kappa, \varrho, d, D).$$

For the multi-Littlewood-Richardson coefficient to be nonzero, it is necessary that all $\mu^i \subseteq (5d-1, 1)$, so each μ^i is either a single row or a hook $(\widehat{\varrho}_i \cdot i \cdot d - 1, 1)$. But $a_{nm-1,1}(n, m) = 0$ and $a_{nm}(n, m) = 1$, so we can assume that the sum has only the summand with $\mu^i = (\widehat{\varrho}_i \cdot i \cdot d)$ and the product of plethysm coefficients is 1. Hence, the multi-Littlewood-Richardson coefficient counts the number of semistandard tableaux of shape $(5d - 1, 1)$ and content $(\mu^1, \ldots, \mu^5)$.

It is instructive to look at all possible $\widehat{\varrho}$: $(1, \widehat{1, 1, 1, 1}) = (5)$, $(2, \widehat{1, 1, 1}) = (3, 1)$, $(\widehat{2, 2, 1}) = (1, 2)$, $(\widehat{3, 1, 1}) = (2, 0, 1)$, $(\widehat{3, 2}) = (0, 1, 1)$, $(\widehat{4, 1}) = (1, 0, 0, 1)$, $(\widehat{5}) = (0, 0, 0, 0, 1)$. We observe that $\widehat{\varrho}$ has exactly two nonzero entries in 5 cases, and only one nonzero entry in 2 cases. There are no semistandard tableaux of shape $(5d-1, 1)$ with only one entry, and there is exactly one semistandard tableaux of shape $(5d - 1, 1)$ with two symbols and fixed content. Hence

$$\text{mult}_\kappa \mathbb{C}[\text{GL}_{d+1}(x_1^d + x_2^d + \cdots + x_{d+1}^d)] = 5.$$

Note that this argument works indeed for all $d \geq 3$, even though for $d = 3$ we do not have $\varrho = (1, 1, 1, 1, 1)$ in the sum (because it has more than $d + 1 = 4$ rows, but its contribution is zero anyway).

We now apply Theorem 4.2 from [IK20], which implies

$$\text{mult}_{\blacksquare + \kappa} \mathbb{C}[\overline{\text{GL}_{d+1}(x_1^d + x_2^d + \cdots + x_{d+1}^d)}] \geq 5. \qquad\square$$

Acknowledgements

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A Calculation tables

We list the partitions λ for which the plethysm coefficient $a := a_\lambda(\delta, d)$ exceeds the multiplicity $b := \text{mult}_\lambda(\mathbb{C}[\text{GL}_{d+1}(x_1 \cdots x_d + x_{d+1}^d)])$. We write $\lambda_{a>b}$. We list λ always with all $d+1$ parts, i.e., with all trailing zeros. λ always has $d\delta$ many boxes. If we list a case (d, δ) and not list (d, δ') with $\delta' < \delta$, then this means that (d, δ') is empty.

$d = 3$, $\delta = 8$:

$(8,8,4,4)_{2>1}$, $(10,6,4,4)_{4>3}$

$d = 4$, $\delta = 6$:

$(6,6,4,4,4)_{1>0}$, $(7,7,5,5,0)_{1>0}$, $(7,7,7,3,0)_{1>0}$, $(8,5,5,3,3)_{1>0}$

$d = 4$, $\delta = 7$:

$(7,7,5,5,4)_{1>0}$, $(7,7,6,5,3)_{1>0}$, $(7,7,7,4,3)_{1>0}$, $(7,7,7,5,2)_{1>0}$, $(7,7,7,7,0)_{1>0}$, $(8,6,6,4,4)_{4>1}$, $(8,7,5,4,4)_{1>0}$, $(8,7,5,5,3)_{2>0}$, $(8,7,6,4,3)_{4>2}$, $(8,7,6,5,2)_{4>1}$, $(8,7,7,3,3)_{3>0}$, $(8,7,7,4,2)_{1>0}$, $(8,7,7,5,1)_{3>0}$, $(8,8,4,4,4)_{4>2}$, $(8,8,5,4,3)_{4>1}$, $(8,8,6,4,2)_{9>4}$, $(8,8,7,3,2)_{3>1}$, $(8,8,7,4,1)_{4>3}$, $(8,8,8,2,2)_{3>2}$, $(9,6,5,4,4)_{3>0}$, $(9,6,5,5,3)_{1>0}$, $(9,6,6,4,3)_{5>3}$, $(9,6,6,5,2)_{4>3}$, $(9,7,4,4,4)_{2>1}$, $(9,7,5,4,3)_{7>2}$, $(9,7,5,5,2)_{5>1}$, $(9,7,6,3,3)_{5>3}$, $(9,7,6,4,2)_{10>5}$, $(9,7,6,5,1)_{6>4}$, $(9,7,7,3,2)_{5>1}$, $(9,7,7,4,1)_{5>2}$, $(9,7,7,5,0)_{2>1}$, $(9,8,4,4,3)_{5>2}$, $(9,8,5,3,3)_{4>1}$, $(9,8,5,4,2)_{11>5}$, $(9,8,5,5,1)_{4>3}$, $(9,8,6,3,2)_{11>6}$, $(9,8,6,4,1)_{12>11}$, $(9,8,7,2,2)_{5>3}$, $(9,8,7,3,1)_{8>6}$, $(9,9,4,3,3)_{3>1}$, $(9,9,4,4,2)_{2>1}$, $(9,9,5,3,2)_{7>5}$, $(9,9,5,4,1)_{6>4}$, $(10,5,5,5,3)_{1>0}$, $(10,6,4,4,4)_{7>2}$, $(10,6,5,4,3)_{6>2}$, $(10,6,5,5,2)_{2>0}$, $(10,6,6,4,2)_{13>8}$, $(10,7,4,4,3)_{8>4}$, $(10,7,5,3,3)_{7>3}$, $(10,7,5,4,2)_{14>6}$, $(10,7,5,5,1)_{6>2}$, $(10,7,6,3,2)_{14>8}$, $(10,7,6,4,1)_{15>13}$, $(10,7,7,2,2)_{1>0}$, $(10,7,7,3,1)_{10>5}$, $(10,8,4,3,3)_{2>1}$, $(10,8,4,4,2)_{17>9}$, $(10,8,5,3,2)_{15>8}$, $(10,8,5,4,1)_{17>14}$, $(10,8,6,2,2)_{17>10}$, $(10,9,4,3,2)_{10>7}$, $(10,9,4,4,1)_{10>9}$, $(10,9,5,2,2)_{10>6}$, $(10,10,4,2,2)_{9>5}$, $(11,5,4,4,4)_{2>1}$, $(11,5,5,4,3)_{3>0}$, $(11,6,4,4,3)_{8>4}$, $(11,6,5,3,3)_{3>2}$, $(11,6,5,4,2)_{13>6}$, $(11,6,5,5,1)_{3>2}$, $(11,6,6,3,2)_{10>9}$, $(11,7,4,3,3)_{6>3}$, $(11,7,4,4,2)_{14>9}$, $(11,7,5,3,2)_{18>9}$, $(11,7,5,4,1)_{18>15}$, $(11,7,6,2,2)_{12>7}$, $(11,7,7,2,1)_{8>7}$, $(11,8,4,3,2)_{17>10}$, $(11,8,5,2,2)_{17>12}$, $(11,9,3,3,2)_{5>3}$, $(11,9,4,2,2)_{12>9}$, $(11,10,3,2,2)_{6>4}$, $(12,4,4,4,4)_{4>3}$, $(12,5,4,4,3)_{4>2}$, $(12,5,5,3,3)_{3>0}$, $(12,5,5,4,2)_{3>1}$, $(12,5,5,5,1)_{1>0}$, $(12,6,4,4,2)_{17>11}$, $(12,6,5,3,2)_{12>8}$, $(12,6,5,4,1)_{13>12}$, $(12,6,6,2,2)_{13>10}$, $(12,7,3,3,3)_{1>0}$, $(12,7,4,3,2)_{17>11}$, $(12,7,5,2,2)_{14>10}$, $(12,8,3,3,2)_{4>3}$, $(12,8,4,2,2)_{23>18}$, $(12,9,3,2,2)_{9>8}$, $(13,5,4,3,3)_{2>0}$, $(13,5,4,4,2)_{8>6}$, $(13,5,5,3,2)_{4>2}$, $(13,5,5,4,1)_{4>3}$, $(13,6,4,3,2)_{13>11}$, $(13,6,5,2,2)_{13>11}$, $(13,7,3,3,2)_{5>3}$, $(13,7,4,2,2)_{16>14}$, $(13,8,3,2,2)_{12>11}$, $(14,5,4,3,2)_{7>5}$, $(15,5,3,3,2)_{1>0}$

$d = 4$, $\delta = 8$:

$(7,7,7,7,4)_{1>0}$, $(8,6,6,6,6)_{2>1}$, $(8,7,6,6,5)_{1>0}$, $(8,7,7,5,5)_{3>0}$, $(8,7,7,6,4)_{1>0}$, $(8,7,7,7,3)_{2>0}$, $(8,8,6,6,4)_{7>1}$, $(8,8,7,5,4)_{3>0}$, $(8,8,7,6,3)_{5>0}$, $(8,8,8,4,4)_{8>2}$, $(8,8,8,5,3)_{2>1}$, $(8,8,8,6,2)_{7>2}$, $(9,6,6,6,5)_{2>1}$, $(9,7,6,5,5)_{3>0}$, $(9,7,6,6,4)_{5>1}$, $(9,7,7,5,4)_{7>0}$, $(9,7,7,6,3)_{6>0}$, $(9,7,7,7,2)_{3>0}$, $(9,8,5,5,5)_{1>0}$, $(9,8,6,5,4)_{14>2}$, $(9,8,6,6,3)_{12>3}$, $(9,8,7,4,4)_{10>1}$, $(9,8,7,5,3)_{18>2}$, $(9,8,7,6,2)_{13>2}$, $(9,8,7,7,1)_{3>0}$, $(9,8,8,4,3)_{11>2}$, $(9,8,8,5,2)_{12>4}$, $(9,8,8,6,1)_{7>4}$, $(9,9,5,5,4)_{6>0}$, $(9,9,6,4,4)_{5>0}$, $(9,9,6,5,3)_{15>3}$, $(9,9,6,6,2)_{5>2}$, $(9,9,7,4,3)_{14>1}$, $(9,9,7,5,2)_{17>3}$, $(9,9,7,6,1)_{7>2}$, $(9,9,7,7,0)_{2>0}$, $(9,9,8,3,3)_{8>1}$, $(9,9,8,4,2)_{8>1}$, $(9,9,8,5,1)_{9>3}$, $(9,9,9,3,2)_{3>1}$, $(9,9,9,4,1)_{3>0}$, $(10,6,6,6,4)_{9>3}$, $(10,7,5,5,5)_{3>0}$, $(10,7,6,5,4)_{15>1}$, $(10,7,6,6,3)_{13>3}$, $(10,7,7,4,4)_{5>0}$, $(10,7,7,5,3)_{19>1}$, $(10,7,7,6,2)_{8>0}$, $(10,7,7,7,1)_{4>0}$, $(10,8,5,5,4)_{7>0}$, $(10,8,6,4,4)_{31>4}$, $(10,8,6,5,3)_{32>5}$, $(10,8,6,6,2)_{29>8}$, $(10,8,7,4,3)_{35>5}$, $(10,8,7,5,2)_{34>6}$, $(10,8,7,6,1)_{18>6}$, $(10,8,8,3,3)_{4>1}$, $(10,8,8,4,2)_{33>9}$, $(10,8,8,5,1)_{15>9}$, $(10,9,5,4,4)_{15>1}$, $(10,9,5,5,3)_{16>1}$, $(10,9,6,4,3)_{39>6}$, $(10,9,6,5,2)_{38>8}$, $(10,9,6,6,1)_{16>9}$, $(10,9,7,3,3)_{21>5}$, $(10,9,7,4,2)_{43>8}$, $(10,9,7,5,1)_{28>9}$, $(10,9,8,3,2)_{24>7}$, $(10,9,8,4,1)_{24>10}$, $(10,9,9,2,2)_{2>0}$,

$(10,9,9,3,1)_{8>3}$, $(10,10,4,4,4)_{12>2}$, $(10,10,5,4,3)_{18>3}$, $(10,10,5,5,2)_{7>0}$, $(10,10,6,3,3)_{8>2}$,
$(10,10,6,4,2)_{42>10}$, $(10,10,6,5,1)_{18>7}$, $(10,10,6,6,0)_{11>10}$, $(10,10,7,3,2)_{23>6}$, $(10,10,7,4,1)_{26>12}$,
$(10,10,8,2,2)_{17>5}$, $(10,10,8,3,1)_{13>9}$, $(10,10,9,2,1)_{6>4}$, $(11,6,6,5,4)_{9>1}$, $(11,6,6,6,3)_{10>4}$,
$(11,7,5,5,4)_{11>0}$, $(11,7,6,4,4)_{22>3}$, $(11,7,6,5,3)_{31>4}$, $(11,7,6,6,2)_{19>6}$, $(11,7,7,4,3)_{25>3}$,
$(11,7,7,5,2)_{25>2}$, $(11,7,7,6,1)_{11>2}$, $(11,7,7,7,0)_{2>0}$, $(11,8,5,4,4)_{26>3}$, $(11,8,5,5,3)_{23>2}$,
$(11,8,6,4,3)_{60>11}$, $(11,8,6,5,2)_{58>13}$, $(11,8,6,6,1)_{24>13}$, $(11,8,7,3,3)_{26>4}$, $(11,8,7,4,2)_{64>14}$,
$(11,8,7,5,1)_{40>15}$, $(11,8,8,3,2)_{28>9}$, $(11,8,8,4,1)_{30>17}$, $(11,9,4,4,4)_{11>1}$, $(11,9,5,4,3)_{45>6}$,
$(11,9,5,5,2)_{33>5}$, $(11,9,6,3,3)_{36>8}$, $(11,9,6,4,2)_{78>19}$, $(11,9,6,5,1)_{46>20}$, $(11,9,7,3,2)_{57>14}$,
$(11,9,7,4,1)_{58>24}$, $(11,9,8,2,2)_{20>7}$, $(11,9,8,3,1)_{37>21}$, $(11,9,9,2,1)_{9>5}$, $(11,10,4,4,3)_{21>5}$,
$(11,10,5,3,3)_{20>4}$, $(11,10,5,4,2)_{52>12}$, $(11,10,5,5,1)_{20>7}$, $(11,10,6,3,2)_{56>16}$, $(11,10,6,4,1)_{56>29}$,
$(11,10,7,2,2)_{30>9}$, $(11,10,7,3,1)_{46>26}$, $(11,10,8,2,1)_{25>20}$, $(11,11,4,3,3)_{10>2}$, $(11,11,4,4,2)_{10>3}$,
$(11,11,5,3,2)_{26>7}$, $(11,11,5,4,1)_{23>12}$, $(11,11,6,2,2)_{13>5}$, $(11,11,6,3,1)_{30>18}$, $(11,11,7,2,1)_{19>15}$,
$(12,6,5,5,4)_{4>0}$, $(12,6,6,4,4)_{21>3}$, $(12,6,6,5,3)_{14>3}$, $(12,6,6,6,2)_{17>8}$, $(12,7,5,4,4)_{19>1}$,
$(12,7,5,5,3)_{22>1}$, $(12,7,6,4,3)_{49>10}$, $(12,7,6,5,2)_{46>9}$, $(12,7,6,6,1)_{17>10}$, $(12,7,7,3,3)_{23>3}$,
$(12,7,7,4,2)_{32>5}$, $(12,7,7,5,1)_{26>6}$, $(12,8,4,4,4)_{25>5}$, $(12,8,5,4,3)_{56>8}$, $(12,8,5,5,2)_{32>5}$,
$(12,8,6,3,3)_{32>7}$, $(12,8,6,4,2)_{109>29}$, $(12,8,6,5,1)_{54>27}$, $(12,8,7,3,2)_{62>17}$, $(12,8,7,4,1)_{65>31}$,
$(12,8,8,2,2)_{30>13}$, $(12,8,8,3,1)_{27>20}$, $(12,9,4,4,3)_{33>6}$, $(12,9,5,3,3)_{35>7}$, $(12,9,5,4,2)_{80>18}$,
$(12,9,5,5,1)_{32>11}$, $(12,9,6,3,2)_{88>28}$, $(12,9,6,4,1)_{88>45}$, $(12,9,7,2,2)_{41>14}$, $(12,9,7,3,1)_{71>40}$,
$(12,9,8,2,1)_{34>28}$, $(12,10,4,3,3)_{14>4}$, $(12,10,4,4,2)_{52>16}$, $(12,10,5,3,2)_{63>18}$, $(12,10,5,4,1)_{62>32}$,
$(12,10,6,2,2)_{60>23}$, $(12,10,6,3,1)_{71>48}$, $(12,10,7,2,1)_{50>41}$, $(12,11,3,3,3)_{2>0}$, $(12,11,4,3,2)_{32>11}$,
$(12,11,4,4,1)_{25>16}$, $(12,11,5,2,2)_{32>14}$, $(12,11,5,3,1)_{46>31}$, $(12,11,6,2,1)_{41>38}$, $(12,12,3,3,2)_{3>2}$,
$(12,12,4,2,2)_{19>10}$, $(12,12,4,3,1)_{13>11}$, $(13,5,5,5,4)_{1>0}$, $(13,6,5,4,4)_{15>1}$, $(13,6,5,5,3)_{9>0}$,
$(13,6,6,4,3)_{26>7}$, $(13,6,6,5,2)_{24>8}$, $(13,7,4,4,4)_{17>4}$, $(13,7,5,4,3)_{45>7}$, $(13,7,5,5,2)_{28>3}$,
$(13,7,6,3,3)_{30>8}$, $(13,7,6,4,2)_{73>21}$, $(13,7,6,5,1)_{39>18}$, $(13,7,7,3,2)_{34>7}$, $(13,7,7,4,1)_{36>15}$,
$(13,7,7,5,0)_{12>11}$, $(13,8,4,4,3)_{38>9}$, $(13,8,5,3,3)_{33>6}$, $(13,8,5,4,2)_{88>23}$, $(13,8,5,5,1)_{32>13}$,
$(13,8,6,3,2)_{91>31}$, $(13,8,6,4,1)_{91>55}$, $(13,8,7,2,2)_{43>17}$, $(13,8,7,3,1)_{65>41}$, $(13,9,4,3,3)_{25>6}$,
$(13,9,4,4,2)_{55>18}$, $(13,9,5,3,2)_{85>28}$, $(13,9,5,4,1)_{78>41}$, $(13,9,6,2,2)_{62>26}$, $(13,9,6,3,1)_{94>67}$,
$(13,9,7,2,1)_{59>50}$, $(13,10,3,3,3)_{4>1}$, $(13,10,4,3,2)_{55>21}$, $(13,10,4,4,1)_{46>33}$, $(13,10,5,2,2)_{57>24}$,
$(13,10,5,3,1)_{75>54}$, $(13,10,6,2,1)_{69>68}$, $(13,11,3,3,2)_{15>6}$, $(13,11,4,2,2)_{32>17}$,
$(13,11,4,3,1)_{44>37}$, $(13,12,3,2,2)_{13>8}$, $(13,13,2,2,2)_{1>0}$, $(14,5,5,4,4)_{2>0}$, $(14,5,5,5,3)_{3>0}$,
$(14,6,4,4,4)_{18>4}$, $(14,6,5,4,3)_{26>4}$, $(14,6,5,5,2)_{11>1}$, $(14,6,6,3,3)_{8>4}$, $(14,6,6,4,2)_{45>17}$,
$(14,6,6,5,1)_{17>13}$, $(14,7,4,4,3)_{31>9}$, $(14,7,5,3,3)_{29>6}$, $(14,7,5,4,2)_{63>17}$, $(14,7,5,5,1)_{24>8}$,
$(14,7,6,3,2)_{63>23}$, $(14,7,6,4,1)_{62>40}$, $(14,7,7,2,2)_{14>4}$, $(14,7,7,3,1)_{38>21}$, $(14,8,4,3,3)_{18>4}$,
$(14,8,4,4,2)_{66>24}$, $(14,8,5,3,2)_{78>27}$, $(14,8,5,4,1)_{76>47}$, $(14,8,6,2,2)_{70>33}$, $(14,8,6,3,1)_{83>68}$,
$(14,9,3,3,3)_{5>2}$, $(14,9,4,3,2)_{63>26}$, $(14,9,4,4,1)_{52>38}$, $(14,9,5,2,2)_{61>29}$, $(14,9,5,3,1)_{85>65}$,
$(14,10,3,3,2)_{15>6}$, $(14,10,4,2,2)_{57>30}$, $(14,10,4,3,1)_{56>53}$, $(14,11,3,2,2)_{22>14}$, $(14,12,2,2,2)_{11>9}$,
$(15,5,4,4,4)_{6>2}$, $(15,5,5,4,3)_{8>0}$, $(15,5,5,5,2)_{2>0}$, $(15,6,4,4,3)_{22>7}$, $(15,6,5,3,3)_{12>3}$,
$(15,6,5,4,2)_{38>12}$, $(15,6,5,5,1)_{10>4}$, $(15,6,6,3,2)_{31>17}$, $(15,6,6,4,1)_{30>26}$, $(15,7,4,3,3)_{18>5}$,
$(15,7,4,4,2)_{45>20}$, $(15,7,5,3,2)_{57>21}$, $(15,7,5,4,1)_{54>35}$, $(15,7,6,2,2)_{40>20}$, $(15,7,6,3,1)_{57>49}$,
$(15,7,7,2,1)_{25>23}$, $(15,8,3,3,3)_{2>0}$, $(15,8,4,3,2)_{58>26}$, $(15,8,4,4,1)_{49>42}$, $(15,8,5,2,2)_{59>33}$,
$(15,8,5,3,1)_{74>64}$, $(15,9,3,3,2)_{19>9}$, $(15,9,4,2,2)_{51>32}$, $(15,10,3,2,2)_{28>19}$, $(16,4,4,4,4)_{7>4}$,
$(16,5,4,4,3)_{10>3}$, $(16,5,5,3,3)_{6>0}$, $(16,5,5,4,2)_{8>2}$, $(16,5,5,5,1)_{2>0}$, $(16,6,4,3,3)_{7>3}$,
$(16,6,4,4,2)_{36>18}$, $(16,6,5,3,2)_{30>14}$, $(16,6,5,4,1)_{29>22}$, $(16,6,6,2,2)_{27>18}$, $(16,7,3,3,3)_{3>0}$,
$(16,7,4,3,2)_{42>21}$, $(16,7,5,2,2)_{36>22}$, $(16,7,5,3,1)_{54>50}$, $(16,8,3,3,2)_{13>7}$, $(16,8,4,2,2)_{53>37}$,
$(16,9,3,2,2)_{26>20}$, $(17,4,4,4,3)_{5>4}$, $(17,5,4,3,3)_{4>0}$, $(17,5,4,4,2)_{15>9}$, $(17,5,5,3,2)_{8>3}$,
$(17,5,5,4,1)_{8>5}$, $(17,6,4,3,2)_{26>17}$, $(17,6,5,2,2)_{24>19}$, $(17,7,3,3,2)_{10>5}$, $(17,7,4,2,2)_{33>27}$,
$(17,8,3,2,2)_{24>22}$, $(18,4,4,4,2)_{9>8}$, $(18,5,4,3,2)_{11>7}$, $(18,6,3,3,2)_{4>3}$, $(19,5,3,3,2)_{1>0}$

$d = 5$, $\delta = 7$, $\lambda_1 \leq 8$:

$(8,7,7,5,5,3)_{1>0}$, $(8,7,7,6,4,3)_{1>0}$, $(8,7,7,6,5,2)_{1>0}$, $(8,7,7,7,3,3)_{1>0}$, $(8,8,7,5,4,3)_{2>1}$,
$(8,8,7,6,3,3)_{1>0}$, $(8,8,7,6,4,2)_{3>2}$, $(8,8,7,6,5,1)_{2>1}$, $(8,8,7,7,4,1)_{1>0}$