Sensitivity Conjecture Proof

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Abstract

Sensitivity Conjecture Proof of [Hua19] is presented.

1 Sensitivity Conjecture

Let Q_n be the n-dimensional hyper-cube graph, whose vertex set consists of vectors in $\{-1,1\}^n$, and two vectors are adjacent if they differ in exactly one coordinate. For an undirected graph G, we use the standard graph-theoretic notations $\Delta(G)$ for its maximum degree, and $\lambda_1(G)$ for the largest eigenvalue of its adjacency matrix.

For $x \in \{-1,1\}^n$ and a subset S of indices from $[n]$, we denote by x^S the binary vector x with all indices in S flipped. For $f : \{-1,1\}^n \longrightarrow \{-1,1\}$, the local sensitivity $s(f,x)$ on the input x is defined as the number of indices i, such that $f(x) \neq f(x^{\{i\}})$, and the sensitivity $s(f)$ of f is $\max_x s(f,x)$. The local block sensitivity $bs(f,x)$, is the maximum number of disjoint blocks $B_1,\ B_2,\ \ldots,\ B_n$ of $[n]$, such that for each B_i, $f(x) \neq f(x^{B_i})$. Similarly, the block sensitivity $bs(f)$ of f is $\max_x bs(f,x)$. Although seemingly unnatural, the block sensitivity is known to be polynomially related to many other complexity measures, including the certificate complexity, the decision tree complexity, the quantum query complexity, and the degree of the Boolean function (as real polynomials). Obviously $bs(f) \geq s(f)$.

Conjecture 1.1 (Sensitivity Conjecture)**.** *For every Boolean function f, $bs(f) \leq \mathsf{poly}(s(f))$.*

2 Proof

Recall that Q_n denotes the n-dimensional hyper-cube graph. For an induced graph G of Q_n, let $Q_n - G$ denote the subgraph of Q_n induced on the vertex set $V(Q_n) \setminus V(G)$. For an induced graph G of Q_n, let $\Gamma(G) \overset{\text{def}}{=\!=} \max(\Delta(G), \Delta(Q_n - G))$. The degree of a Boolean function f, denoted by $\deg(f)$, is the degree of the unique multi-linear real polynomial that represents f. Gotsman and Linial [GL92] proved the following remarkable result.

Theorem 2.1 (Gotsman and Linial [GL92])**.** *Consider the following two statements.*

1. *For any induced sub-graph G of Q_n with $|V(G)| \neq 2^{n-1}$, we have $\Gamma(G) \geq \sqrt{n}$.*
2. *For any Boolean function f, we have $s(f) \geq \sqrt{\deg(f)}$.*

Then $(1) \iff (2)$.

Proof. We first transform (1) into a statement concerning Boolean functions: Associate with the sub-graph G a Boolean function g such that $g(x) = 1$ iff $x \in V(G)$. Note that $\deg_G(x) = n - s(g,x)$ for $x \in V(G)$. We also have that $\deg_{Q_n - G}(x) = n - s(g,x)$ for $x \notin V(G)$. Therefore (1) can be restated as (A): For any Boolean function g, $\mathbb{E}(g) \neq 0$ implies that $\exists x : s(g,x) \leq n - \sqrt{n}$.

Now consider the following statement (B): For any Boolean function f with $\deg(f) = n$ we have $s(f) \geq \sqrt{n}$. Clearly, (2) implies (B). We prove that (B) implies (2). Let f be a Boolean function of degree d. Fix a monomial of degree d of the representing polynomial of f. Without loss of generality we may assume the monomial is $x_1 x_2 \ldots x_d$. Define $g(x_1, x_2, \ldots, x_d) \overset{\text{def}}{=\!=} f(x_1, x_2, \ldots, x_d, 1, \ldots, 1)$. Then, $s(f) \geq s(g) \geq \sqrt{d}$, as desired. Thus (2) is equivalent to (B).

Therefore we have the following equivalences.

$$(1) \iff (A)$$
$$(2) \iff (B)$$

Proof of $(A) \implies (B)$: suppose (B) is false. Thus there exists a Boolean function f with $s(f) < \sqrt{n}$ with $\deg(f) = n$. Consider the function $g(x) \stackrel{\text{def}}{=\!=} f(x)p(x)$, where $p(x) = \prod_{i \in [n]} x_i$ is the parity function. Observe that Fourier coefficient $\hat{f}(I) = \hat{g}([n] \setminus I)$ for any subset $I \subseteq [n]$. Since $\deg(f) = n$, we get that $\mathbb{E}(g) = \hat{g}(\emptyset) \neq 0$. Furthermore, $s(g,x) = n - s(f,x)$. This implies that $s(g,x) > n - \sqrt{n}$. Thus $(A) \implies (B)$.

Proof of $(B) \implies (A)$: Assume that $\forall x, s(g,x) > n - \sqrt{n}$. This implies that $s(f) < \sqrt{n}$. By (B), $\deg(f) < n$, which is equivalent to $\hat{f}([n]) = \hat{g}(\emptyset) = \mathbb{E}(g) = 0$ contradicting the premise. □

[NS92] showed that $bs(f) \leq 2(\deg(f))^2$. Thus if we can prove statement (1) of Theorem 2.1 then we obtain that $bs(f) \leq 2(s(f))^4$. This would imply the Sensitivity conjecture (Theorem 1.1). Part (1) of Theorem 2.1 was proved in [Hua19]. We prove it in the rest of the write-up. We first recall the following results. Recall that if polynomials $f(x)$ and $g(x)$ have all real roots $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_{n-1}$ then we say that f and g interlace if and only if $r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq s_{n-1} \leq r_n$.

Theorem 2.2 ([Rah+02]). *The roots of polynomials f, g interlace if and only if the linear combinations $f + \alpha g$ have all real roots for all $\alpha \in \mathbb{R}$.*

Corollary 2.1 ([Fis05]). *If A is a Hermitian matrix, and B is a principle sub-matrix of A, then the eigenvalues of B interlace the eigenvalues of A.*

Proof. Choose $\alpha \in \mathbb{R}$, partition A as:

$$A = \begin{bmatrix} B & c \\ c^* & d \end{bmatrix}.$$

and consider the following equation that follows from linearity of the determinant:

$$\begin{vmatrix} B - xI & c \\ c^* & d - x + \alpha \end{vmatrix} = \begin{vmatrix} B - xI & c \\ c^* & d - x \end{vmatrix} + \begin{vmatrix} B - xI & c \\ 0 & \alpha \end{vmatrix}.$$

Since the matrix on the left hand side is the characteristic polynomial of a Hermitian matrix, $|A - xI| + \alpha |B - xI|$ has all real roots for any α, and hence the eigenvalues interlace. □

Lemma 2.1. *Let us consider a block matrix M of size $(n+m) \times (n+m)$ of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Here A, B, C, D are $n \times n, n \times m, m \times n, m \times m$ respectively. If D is invertible then

$$\det(M) = \det(A - BD^{-1}C)\det(D).$$

If $m = n$ and if C, D commute then $\det(M) = \det(AD - BC)$.

Theorem 2.3 (Cauchy's Interlace Theorem, [Fis05]). *Let A be a symmetric $n \times n$ matrix, and B be a $m \times m$ principal sub-matrix (obtained by deleting the same set of rows and columns from A) of A, for some $m < n$. If the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and the eigenvalues of B are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, then for all $1 \leq i \leq m$ we have that $\lambda_i \geq \mu_i \geq \lambda_{i+n-m}$.*

Proof. We just need to prove the case when $m = n - 1$, because then the claim follows from induction on $n - m$. The case $m = n - 1$ follows from Corollary 2.1. □

Lemma 2.2. *We define a sequence of symmetric square matrices iteratively as follows,*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then A_n is a $2^n \times 2^n$ matrix whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.

Proof. First notice that $A_n^2 = nI$ (by induction on n). The characteristic polynomial $p_n(x)$ of A_n is:

$$p_n(x) \xlongequal{\text{def}} \det(xI - A_n) = \det\begin{pmatrix} xI - A_{n-1} & I \\ I & xI + A_{n-1} \end{pmatrix}.$$

We can use Lemma 2.1 with I and $xI + A_{n-1}$ commuting. We obtain that:

$$\begin{aligned} p_n(x) &= \det((xI - A_{n-1})(xI + A_{n-1}) - I). \\ &= \det((x^2-1)I - A_{n-1}^2) \\ &= \det((x^2-1)I - (n-1)I) \\ &= (x^2 - n)^{2^{n-1}} \end{aligned}$$

$\square$

Lemma 2.3. *Suppose H is an m-vertex undirected graph, and A is a symmetric matrix whose entries are in $\{\pm 1, 0\}$ and whose rows and columns are indexed by $V(H)$, and whenever u and v are non-adjacent in H, $A(u,v) = 0$. Then*

$$\Delta(H) \geq \lambda_1 \xlongequal{\text{def}} \lambda_1(A).$$

Proof. Suppose $\vec{v}$ is the eigen-vector corresponding to λ_1. Then $\lambda_1 \vec{v} = A\vec{v}$. Without loss of generality, assume v_1 is the coordinate of $\vec{v}$ that has the largest absolute value. Then

$$|\lambda_1 v_1| = |(A\vec{v})_1| = \left|\sum_{j=1}^{m} A_{1,j} v_j\right| = \left|\sum_{j \sim 1} A_{1,j} v_j\right| \leq \sum_{j\sim 1} |A_{1,j}|\,|v_1| \leq \Delta(H)\,|v_1|\,.$$

Therefore $|\lambda_1| \leq \Delta(H)$. $\square$

Theorem 2.4. *For every integer $n \geq 1$, let H be an arbitrary $(2^{n-1}+1)$-vertex induced sub-graph of Q_n, then $\Delta(n) \geq \sqrt{n}$.*

Proof. Let A_n be the sequence of matrices defined in Lemma 2.2. Note that the entries of A_n are in $\{\pm 1, 0\}$. By the iterative construction of A_n, it is not hard to see that when changing every -1-entry of A_n to 1, we get exactly the adjacency matrix of Q_n, and thus A_n and Q_n satisfy the conditions in Lemma 2.3. For example, we may let the upper-left and lower-right blocks of A_n correspond to the two $(n-1)$-dimensional sub-cubes of Q_n, and the two identity blocks correspond to the perfect matching connecting these two sub-cubes. Therefore, a $(2^{n-1}+1)$-vertex induced sub-graph H of Q_n and the principal sub-matrix A_H of A_n naturally induced by H also satisfy the conditions of Lemma 2.3. As a result,

$$\Delta(H) \geq \lambda_1(A_H).$$

On the other hand, from Lemma 2.2, the eigenvalues of A_n are known to be

$$\sqrt{n}, \cdots, \sqrt{n}, -\sqrt{n}, \cdots, -\sqrt{n}.$$

Note that A_H is a $(2^{n-1}+1) \times (2^{n-1}+1)$ sub-matrix of the $2^n \times 2^n$ matrix A_n. By Cauchy's Interlace Theorem (Theorem 2.3),

$$\lambda_1(A_H) \geq \lambda_{1+2^n-(2^{n-1}+1)}(A_n) = \lambda_{2^{n-1}}(A_n) = \sqrt{n}.$$

Combining the two inequalities we just obtained ans using Lemma 2.3, we have $\Delta(H) \geq \sqrt{n}$, completing the proof of our theorem. $\square$

Now let us see an alternative proof due to shalev ben-david.

Proof. Let S be the $\sqrt{n}$-eigenspace of the matrix A_n. Then S has dimension 2^{n-1}. Consider a large principal sub-matrix B of A_n. We wish to lower bound its spectral norm (maximum eigenvalue). This is the same as maximizing over vectors x with norm 1. But it's not hard to see that this maximum is the same as the maximum of over unit vectors x that have a 0 entry on indices corresponding to rows/columns that are not in B. Let L by the subspace of all such vectors with 0 entries on those indices. Then L has dimension at least $2^{n-1}+1$, since B uses at least that many rows/columns. It follows that the intersection of L with S must have dimension at least 1. Thus there is a unit vector x that is in both L and S, meaning it has 0's on entries corresponding to rows/columns not in B but it is a $\sqrt{n}$-eigenvector of A_n. This gives a lower bound of $\sqrt{n}$ on the spectral norm of B, as desired. $\square$

3 Inertia

For a Hermitian matrix H, we use $n^+(H)$ to denote the number of positive eigen values of H, $n^-(H), n^0(H)$ are also defined similarity.

Definition 3.1 ([LP15]). For a graph G, $\alpha_q(G)$ is the maximum integer $t \in \mathbb{N}$ for which there exist positive semi-definite matrices $\rho, \rho_i^u \in \mathcal{S}_+^d$ for $i \in [t], u \in V(G)$ (for some $d \geq 1$) satisfying the following conditions:

$$\begin{aligned}
\langle \rho, \rho \rangle &= 1 \\
\sum_{u \in V(G)} \rho_i^u &= \rho && (\forall i \in [t]) \\
\langle \rho_i^u, \rho_j^v \rangle &= 0 && (\forall i \neq j \in [t], \forall \{u, v\} \in E(G) \text{ or } u = v) \\
\langle \rho_i^u, \rho_i^v \rangle &= 0 && (\forall i \in [t], \forall u \neq v \in V(G))
\end{aligned}$$

Here $\mathcal{S}_+^d$ is the cone of $d \times d$ positive semi-definite matrices for an arbitrary $d \geq 1$.

Theorem 3.1 ([WEA19]). *For all graphs G and all Hermitian matrices H,*

$$\alpha(G) \leq \alpha_q(G) \leq \alpha_p(G) \leq n^0(H \circ A_G) + \min\{n^+(H \circ A_G) + n^-(H \circ A_G)\}.$$

Here $\circ$ denotes the Hadamard product (also called the Schur or entry-wise product).

$\alpha_p(G)$ is the projective packing number of G. There exist graphs G for which there is an exponential separation between the independence number $\alpha(G)$ and $\alpha_q(G)$ [MR16].

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